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Proceedings of a symposium, held at Durham in September 1977, on 'Homological and combinational techniques in group theory'

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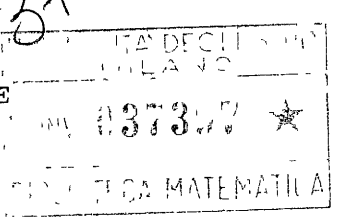

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Contents

Preface

The reader may distinguish three principal themes in this volume. There is the direct development of homological methods, interlocking neatly with the Euler characteristic theory on one side, and finiteness questions on the other. There is the theory of groups acting on trees, including that of amalgamated free products and HNN groups, and also the Stallings structure theorem. Finally, but at present still in a rudimentary state, there is the technique of relation modules.

In contrast there is a need for examples, general enough to test ideas, but explicit enough to make detailed calculations. Much the most interesting at present are arithmetic and related groups; the study of these was an auxiliary theme.

Thanks are due to the LMS for backing the conference, to the SRC for money to run it, to David Johnson for much work on the organisation, to the staff at Grey College for providing an agreeable background and, of course, to the participants for their contributions.

C. T. C. Wall

Introduction

There have been many instances of the use of topological ideas in connexion with infinite group theory; the most obvious being, perhaps, the cohomology of groups and the most spectacular, Stallings' theorem on the structure of groups with infinitely many ends. Although the pioneering papers were somewhat isolated from each other, there have been signs in recent years that the techniques are being brought together into a new branch of group theory. The object of the conference was to bring together the main people active in this area, and these proceedings are intended to give a general view of these developments.

The papers in this volume were invited from the participants in the above symposium. Six main speakers presented surveys of different areas in three or four lectures; written versions of these form the first five items in the contents (the notes by Scott and Wall contain much of the material from Stallings' lectures as well as that from mine). The other items do not all correspond closely with talks given at the symposium and in several cases present work done subsequently.

1 · Traces and Euler characteristics

HYMAN BASS

Columbia University

Let A be a ring. I shall write $\mathcal{P}(A)$ for the category of finitely generated projective right A-modules and $K_0(A)$ for its Grothendieck group. When A is an algebra over some commutative ring k let $\mathcal{R}_k(A)$ denote the category of right A-modules M such that $M \in \mathcal{P}(k)$, the category of 'k representations' of A, and let $R_k(A)$ denote its Grothendieck group.

I shall be mainly concerned with $\mathcal{P}(A)$ in the case when $A = kG$, the group algebra of a group G, and particularly the case when $k = \mathbb{Z}$. This is a subject that barely exists except for some very special classes of groups G, notably finite groups and abelian groups. The following questions indicate the level of our ignorance.

1. Let G be a torsion free group.

(i) Is every $P \in \mathcal{P}(\mathbb{Z}G)$ free?

No in general but there is essentially only one example known [D], Dunwoody's trefoil module. $G = \langle x, y \,|\, x^2 = y^3 \rangle$ is the trefoil group, P is a relation module arising from a presentation of G, and $P \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$.

(ii) Is $K_0(\mathbb{Z}G) \cong \mathbb{Z}$?

No counterexamples are known.

I mention in passing the following classical problem, which turns out to be related to the above questions in certain cases.

(iii) Is $\mathbb{Z}G$ without non trivial 0-divisors?

(iv) (Serre [S]). Suppose that G is of type (FP), i. e. there is a finite resolution

$$0 \to P_n \to \dots \to P_0 \to \mathbb{Z} \to 0$$

with each $P_i \in \mathcal{P}(\mathbb{Z}G)$. Is G then of type (FL)? I. e. can one choose all the P_i to be free?

2. Can one, for a reasonably extensive class of groups G, describe $K_0(\mathbb{Z}G)$ or at least $K_0(\mathbb{Q}G)$ in terms of the finite subgroups of G?

A specific case: Let $G = SL_n(\mathbb{Z})$ with $n \geq 3$. If H is a finite subgroup of G and if $Q \in \mathcal{P}(\mathbb{Z}H)$, we can form the induced module $P = \mathrm{Ind}_H^G(Q) \in \mathcal{P}(\mathbb{Z}G)$. My student David Carter has produced such examples which are non free [C]. In fact he shows, for any odd prime p, that $K_0(\mathbb{Z}SL_p(\mathbb{Z}))$ contains a subgroup of order $\frac{p-1}{2}$. However no one has produced non free projective modules over torsion free subgroups of $SL_n(\mathbb{Z})$. Modules obtained by induction from finite subgroups as above always restrict to free modules over torsion free subgroups.

I propose to discuss here a rank invariant r_P of projective modules $P \in \mathcal{P}(A)$. There are essentially two ways of constructing such invariants:

- Make a base change $A \to B$ so that $P \otimes_A B$ is B-free and count a basis.
- Define $r_P = T_{P/A}(1_P)$ where $T_{P/A}$ is a 'trace function' on $\mathrm{End}_A(P)$.

The first method is that used in commutative algebra, taking for B the various localizations of A. It is sometimes used also for group algebras $A = kG$, using the augmentation $A \to k$.

The second method appears to make sense only over commutative rings, since it is only then that one classically can define the trace of an endomorphism. However Stallings and Hattori introduced a trace in the general case and it is this trace that we shall use.

Conjecture (4.5) below asserts, for any group G and $P \in \mathcal{P}(\mathbb{Z}G)$, that $r_P = r_F$ for some free module F.

The final section 7 discusses various Euler characteristics constructed from such rank functions.

Much of this presentation is a resumé of results in [B].

1. Hattori-Stallings traces (see [B], [H], [St 1])

A denotes a ring; A-modules are understood to be right A-modules and $\mathcal{P}(A)$ denotes the category of those which are finitely generated and projective.

We write

$$T = T_A : A \to T(A) = A/[A, A]$$

for the natural projection to the quotient of A by the additive group $[A, A]$ generated by all commutators $[a, b] = ab - ba$.

Let P be an A-module and $P^* = \mathrm{Hom}_A(P, A)$. If $x \in P$, $a \in A$, $f \in P^*$, we have $T(f(xa)) = T(f(x)a) = T(af(x)) = T((af)(x))$, whence an additive map $P \otimes_A P^* \to T(A)$ sending $x \otimes f$ to $T(f(x))$. On the other hand we have a canonical homomorphism

$$P \otimes_A P^* \to \mathrm{End}_A(P), \quad x \otimes f \mapsto xf : y \mapsto xf(y),$$

which is an isomorphism if and only if $P \in \mathcal{P}(A)$. In this case we view the latter as an identification and so obtain an additive map, called the trace,

$$T_P = T_{P/A} : \mathrm{End}_A(P) \to T(A),$$
$$T_P(x \otimes f) = T_A(f(x)) .$$

By a (finite) coordinate system in P, we mean a finite family (x_i, f_i) in $P \times P^*$ such that $1_P = \sum x_i \otimes f_i$, in other words such that $x = \sum_i x_i f_i(x)$ for all $x \in P$. If $u \in \mathrm{End}_A(P)$, then $u(x_i \otimes f_i) = u(x_i) \otimes f_i$, so $u = u1_P = \sum_i u(x_i) \otimes f_i$ and

$$(1) \qquad T_P(u) = T_A(\sum_i f_i(u(x_i))) .$$

If (x_i) happens to be a free basis of P then (f_i) is the dual basis of P^*, $u_{ji} = f_j(u(x_i))$ defines the matrix of u relative to (x_i), and formula (1) reads: $T_P(u) = T_A(\sum_i u_{ii})$. This shows that when A is commutative and P is free then T_P is the usual trace. We define the rank of P to be the element

$$(2) \qquad r_P = r_{P/A} = T_P(1_P) = T_A(\sum_i f_i(x_i)) \in T(A) .$$

For example

(3) $r_{A^n} = T_A(n)$.

An A-module M is said to be of type (FP) if there is a finite resolution

(4) $0 \to P_n \to \dots \to P_0 \to M \to 0$

for some $n \geq 0$ with each $P_i \in \mathcal{P}(A)$. An endomorphism $u \in \mathrm{End}_A(M)$ can then be lifted to an endomorphism $(u_i \in \mathrm{End}_A(P_i))$ of the resolution (4) and we put

(5) $T_M(u) = \sum_i (-1)^i T_{P_i}(u_i)$.

This gives a well defined map

(6) $T_M : \mathrm{End}_A(M) \to T(A)$

and we define the rank of M to be

(7) $r_M = T_M(1_M) = \sum_i (-1)^i r_{P_i}$.

The above trace maps (6) enjoy the following properties (see [B], [H], [St]).

(1.1) Additivity. Let M be an A-module, M' a submodule, and $M'' = M/M'$. If two of M', M, M'' are of type (FP) so also is the third. Suppose this is the case and that $u \in \mathrm{End}_A(M)$ leaves M' invariant and induces $u' \in \mathrm{End}_A(M')$ and $u'' \in \mathrm{End}_A(M'')$. Then

$$T_M(u) = T_{M'}(u') + T_{M''}(u'') .$$

(When $M = M' \oplus M''$ this corresponds to the matrix formula

$$T_M\begin{pmatrix} u' & * \\ 0 & u'' \end{pmatrix} = T_{M'}(u') + T_{M''}(u'') .)$$

(1.2) Linearity. If $u, v \in \mathrm{End}_A(M)$ then

$$T_M(u + v) = T_M(u) + T_M(v) .$$

Further, if A is a k-algebra for some commutative ring k, then so also is $\mathrm{End}_A(M)$, $T(A)$ is naturally a k-module, and T_M is k-linear,

$$T_M(ut) = T_M(u)t$$

for $t \in k$. Consequently r_M is annihilated by $\mathrm{ann}_k(M)$.

(1.3) Commutativity. Let $M \underset{u'}{\overset{u}{\rightleftarrows}} M'$ be homomorphisms between A-modules of type (FP). Then

$$T_M(u'u) = T_{M'}(uu').$$

(1.4) Universality. Suppose $T'_P : \mathrm{End}_A(P) \to S$ is a collection of maps (defined for $P \in \mathcal{P}(A)$) into an additive group S, which is additive, linear, and commutative in the above sense. Then there is a unique group homomorphism $t : T(A) \to S$ such that $T'_P = t \circ T_P$ for all $P \in \mathcal{P}(A)$. The same applies if we replace $\mathcal{P}(A)$ by the category of all A-modules of type (FP).

(1.5) Functoriality; covariance. A ring homomorphism $\alpha : A \to B$ induces an additive map

$$\alpha_* : T(A) \to T(B),$$
$$T_A(a) \mapsto T_B(\alpha(a)).$$

If $P \in \mathcal{P}(A)$ and $u \in \mathrm{End}_A(P)$ then we have $\alpha_* P = P \otimes_A B \in \mathcal{P}(B)$ and $\alpha_* u = u \otimes_A 1_B \in \mathrm{End}_B(\alpha_* P)$, and

$$T_{\alpha_* P}(\alpha_* u) = \alpha_* T_P(u).$$

In particular $r_{\alpha_* P} = \alpha_* r_P$. If B is a flat left A-module (via α), then these formulae remain valid for all A-modules P of type (FP).

(1.6) Automorphisms. Suppose that α is an automorphism of A. For every A-module M we have the A-module $M^{(\alpha)}$ with M as additive group and scalar operation $x.a = x\alpha(a)$. Then $M \mapsto M^{(\alpha)}$, and $u \mapsto u^{(\alpha)} = u$ for morphisms, is an automorphism of the category of A-modules. The map $x \mapsto x \otimes 1$ is an A-isomorphism from $M^{(\alpha)}$ to

$\alpha_*^{-1}M = M \otimes_{\alpha^{-1}} A$, matching $u \in \mathrm{End}_A(M) = \mathrm{End}_A(M^{(\alpha)})$ with $\alpha_*^{-1}u = u \otimes_{\alpha^{-1}} 1_A \in \mathrm{End}_A(\alpha_*^{-1}M)$. It follows therefore from (1.5) that if M is of type (FP) then

$$T_{M^{(\alpha)}}(u) = \alpha_*^{-1} T_M(u) .$$

In particular

$$r_{M^{(\alpha)}} = \alpha_*^{-1} r_M .$$

(1.7) Contravariance; $\mathrm{Tr}_{B/A}$. If α makes B a right A-module of type (FP) then it does the same to all B-modules M of type (FP), so we can define $T_{M/A}(u)$ for $u \in \mathrm{End}_B(M) \subset \mathrm{End}_A(M)$. The map $(M, u) \mapsto T_{M/A}(u)$ is manifestly additive, linear, and commutative. By universality, therefore, it is of the form

$$T_{M/A}(u) = \mathrm{Tr}_{B/A}(T_{M/B}(u))$$

for a unique homomorphism

$$\mathrm{Tr}_{B/A} : T(B) \to T(A) .$$

2. Characters

Let k be a commutative ring and let A be a k-algebra. If M is an A-module and $a \in A$, the endomorphism $a_M : x \mapsto xa$ of M is k-linear. Let $\mathcal{R}_k(A)$ denote the category of A-modules M which are finitely generated and projective as k-modules. If $M \in \mathcal{R}_k(A)$ we have its character

$$\chi_M : A \to k$$
$$a \mapsto T_{M/k}(a_M) .$$

It is a k-linear map vanishing on $[A, A]$, so we may also view χ_M as an element of $\mathrm{Hom}_k(T(A), k)$.

The additive functor $H : P \mapsto \mathrm{Hom}_A(P, M)$ sends $\mathcal{P}(A)$ to $\mathcal{P}(k)$. If $P \in \mathcal{P}(A)$ and $u \in \mathrm{End}_A(P)$, we have $T_{H(P)/k}(H(u)) \in k$, which is clearly

additive, linear, and commutative in (P, u), whence a homomorphism $\chi : T(A) \to k$ such that $T_{H(P)/k}(H(u)) = \chi(T_{P/A}(u))$. When $P = A$ and $u(x) = ax$ we have an isomorphism $(H(P), H(u)) \cong (M, a_M)$, whence $\chi = \chi_M$. Explicitly,

(2.1) Proposition. If $M \in \mathcal{R}_k(A)$, $P \in \mathcal{P}(A)$, and $u \in \mathrm{End}_A(P)$ then $\mathrm{Hom}_A(P, M) \in \mathcal{P}(k)$ and

$$T_{\mathrm{Hom}_A(P, M)/k}(\mathrm{Hom}_A(u, M)) = \chi_M(T_{P/A}(u)).$$

In particular when $u = 1_P$ we have

$$r_{\mathrm{Hom}_A(P, M)/k} = \chi_M(r_{P/A}) .$$

(2.2) Proposition. Suppose that A is a finitely generated projective k-module. Then every $P \in \mathcal{P}(A)$ is likewise and, if $r_{P/A} = T_A(a)$, we have

$$\chi_P(b) = T_{A/k}(L_a \circ R_b) = T_{A/k}(x \mapsto axb) .$$

In fact let $a_i, f_i : A \to k$ be a finite k-coordinate system of A and let $x_j, g_j : P \to A$ be a finite A-coordinate system of P. If $x \in P$ then $\sum_{i,j} x_j a_i f_i(g_j(x)) = \sum_j x_j g_j(x) = x$, so $x_j a_i$, $f_i g_j : P \to k$ is a k-coordinate system of P. Hence $\chi_P(b) = T_{P/k}(b_P) = \sum_{i,j} f_i g_j(b_P(x_j a_i)) =$ $\sum_{i,j} f_i(g_j(x_j a_i b)) = \sum_i f_i(a a_i b)$ (where $a = \sum_j g_j(x_j)$, so $r_P = T_A(a)$ $= T_{A/k}(x \mapsto axb)$.

3. Group algebras

Let $A = kG$, the group algebra of a group G over a commutative ring k. The k-module $[A, A]$ is generated by the commutators

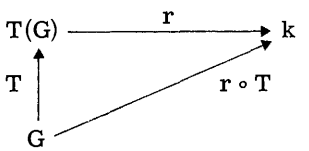
$$[s, t] = st - ts = sus^{-1} - u = [su, s^{-1}]$$

where $s, t, u \in G$ and $u = ts$. Thus $T(s) = T(t)$ in $T(kG)$ if and only if s and t are conjugate in G. We shall thus identify $T(s)$ (or $T_G(s)$) with the G-conjugacy class of s. These classes constitute a k-basis

$T(G)$ of $T(kG)$. If $r \in T(kG)$ we thus have

$$r = \sum_{\tau \in T(G)} r(\tau) \cdot \tau,$$

a notation that interprets r as a function



with finite support $\mathrm{supp}(r) = \{\tau \in T(G) \mid r(\tau) \neq 0\}$. We shall sometimes confuse r with the central function $r \circ T$, writing $r(s)$ for $r(T(s))$ if $s \in G$. For any function f on G we define $\bar{f}$ by $\bar{f}(s) = f(s^{-1})$.

(3.1) Proposition (Hattori [H]). Let G be a finite group and let $P \in \mathcal{P}(kG)$. Then

$$\chi_P(s) = |Z_G(s)| \cdot r_P(s^{-1})$$

for $s \in G$. For $s = 1$ this gives

$$r_{P/k} = |G| \cdot r_P(1).$$

In fact let $a = \sum_{s \in G} a_s s \in kG$ be such that $r_P = T(a)$. According to Proposition (2.2) we have $\chi_P(s) = T_{kG/k}(x \mapsto axs) = \sum_{t \in G} a_t T_{kG/k}(x \mapsto txs)$. Now $x \mapsto txs$ permutes the k-basis G of kG so its trace is the number of $x \in G$ such that $txs = x$, i.e. such that $t = xs^{-1}x^{-1}$. This number is 0 if $t \notin T(s^{-1})$, and $|Z_G(s)|$ if $t \in T(s^{-1})$. Thus $\chi_P(s) = \sum_{t \in T(s^{-1})} a_t \cdot |Z_G(s)| = r_P(s^{-1}) \cdot |Z_G(s)|$.

(3.2) Corollary. If $|G|$ is invertible in k then $r_P = T(a_P)$ where $a_P = |G|^{-1} \sum_{s \in G} \chi_P(s^{-1})s$, an element of the center of kG.

(3.3) Corollary. If k is an integral domain in which no prime divisor of $|G|$ is invertible then 0 and 1 are the only idempotents in kG.

If $\mathrm{char}(k) = p > 0$ then G is a p-group so, if F is the field of fractions of k, then FG is a local ring; whence the corollary.

Suppose $p = 0$ and let $e \neq 0$ be an idempotent in kG. Put $P = ekG$. Then $r_{P/k} = \chi_P(1) = |G| \cdot r_P(1)$, so $r_{P/k}/|G|$ belongs to $k \cap \mathbb{Q}$ and our hypothesis implies that it is an integer. But if $|G|$ divides $r_{P/k}$ and P is a direct summand of kG we must have $P = kG$, so $e = 1$.

4. Subgroups of finite index

Let H be a subgroup of finite index in G. Then

$$kG = \bigoplus_{s \in G/H} skH ,$$

a free kH-module with basis a set of representatives of the cosets G/H. Therefore we have a k-linear map (see (1.7))

$$Tr = Tr_{kG/kH} : T(kG) \to T(kH)$$

defined by $Tr(T_G(a)) = T_{kG/kH}(L_a : x \mapsto ax)$ for $a \in kG$. If $t \in G$ then L_t permutes the direct summands skH above, and $tskH = skH$ if and only if $s^{-1}ts \in H$, in which case $L_t(s) = s \cdot (s^{-1}ts)$. Therefore

$$(1) \qquad Tr(T_G(t)) = \sum_{\substack{s \in G/H \\ s^{-1}ts \in H}} T_H(s^{-1}ts).$$

Let $\tau = T_G(t)$. Then (1) shows that

$$(2) \qquad Tr(\tau) = \sum_{\substack{\sigma \in T(H) \\ \sigma \subset \tau}} z_\sigma \cdot \sigma$$

where z_σ is the number of $s \in G/H$ such that $s^{-1}ts \in \sigma$. If s_0 has this property then s_1 does also if and only if $s_1 \in Z_G(t)s_0H$, so z_σ is the number of H-cosets in the double coset $Z_G(t)s_0H$. This is the index in $Z_G(t)$ of $Z_G(t) \cap s_0Hs_0^{-1} = Z_{s_0Hs_0^{-1}}(t)$, so

$$(3) \qquad z_\sigma = [Z_G(s) : Z_H(s)]$$

for any $s = s_0^{-1}ts_0 \in \sigma$. Suppose that $r = \sum_{\tau \in T(G)} r(\tau)\tau \in T(kG)$. Then

$\mathrm{Tr}(r) = \sum_{\tau \in T(G)} r(\tau) \sum_{\substack{\sigma \in T(H) \\ \sigma \subset \tau}} z_\sigma \cdot \sigma = \sum_{\sigma = T_H(s) \in T(H)} r(s) \cdot z_\sigma \sigma$. In other words, for $s \in H$ we have

(4) $\mathrm{Tr}(r)(s) = r(s) \cdot [Z_G(s) : Z_H(s)]$.

If M is a kG-module of type (FP), hence likewise as kH-module, and if $u \in \mathrm{End}_{kG}(M)$, we have $T_{M/kH}(u) = \mathrm{Tr}(T_{M/kG}(u))$. In case $u = 1_M$ we thus have from (4):

(5) $r_{M/H}(s) = r_{M/G}(s) \cdot [Z_G(s) : Z_H(s)]$

for $s \in H$. When $s = 1$ this becomes

(6) $r_{M/H}(1) = r_{M/G}(1) \cdot [G : H]$.

(4.1) Theorem. Let G be a finite group and let k be an integral domain in which no prime divisor of $|G|$ is invertible. Let $P \in \mathcal{P}(kG)$ and let n denote the rank of the k-module $P \otimes_{kG} k$. Then

(7) $r_P = r_{(kG)^n}$ $(= T_{kG}(n))$.

Since $n = \sum_{\tau \in T(G)} r(\tau)$, the theorem is equivalent to the assertion that

(8) $r_P(s) = 0$ for $s \neq 1$ in G.

Let $s \in G$ and put $H = \langle s \rangle$. Then $r_{P/H}(s) = r_{P/G}(s) \cdot [Z_G(s) : Z_H(s)]$, and the last factor is $\neq 0$ in k, by assumption. Thus it suffices to prove the theorem for the abelian group H. But then $r_{P/H}$ lies in the subring of kH generated by all idempotents. By Corollary (3.3) above, this subring is the prime subring, whence $r_{P/H}(s) = 0$ if $s \neq 1$.

(4.2) Corollary (Swan). Let F be the field of fractions of k. Then $P \otimes_k F \cong (FG)^n$.

If $\mathrm{char}(k) = p > 0$ then G is a p-group, so FG is a local ring, and $P \otimes_k F$ is FG-free. If $p = 0$ then FG is semi-simple and it suffices to show that $\chi_P = \chi_{(kG)^n}$. In view of Proposition (3.1), this

follows from (4.1).

(4.3) Corollary. Let k be a subring of $\mathbb{C}$ such that $k \cap \mathbb{Q} = \mathbb{Z}$. Let G be a group and $P \in \mathcal{P}(kG)$. If G is residually finite then

$$r_P(1) = \sum_{\tau \in T(G)} r_P(\tau) \qquad (= r_{P \otimes_{kG} k/k}).$$

Let $\pi : G \to G'$ be the projection to a finite quotient chosen so that if $r_P(s) \neq 0$ and $s \neq 1$ then $\pi(s) \neq 1$. Since $\mathrm{supp}(r_P)$ is finite, such a π exists. Let $P' = P \otimes_{kG} kG' \in \mathcal{P}(kG')$. By Theorem (4.1), $r_{P'}(s') = 0$ for $s' \neq 1$ in G'. Hence

$$0 = \sum_{\substack{\tau' \in T(G') \\ \tau' \neq 1}} r_{P'}(\tau') = \sum_{\substack{\tau \in T(G) \\ \pi(\tau) \neq 1}} r_P(\tau) = \sum_{\substack{\tau \in T(G) \\ \tau \neq 1}} r_P(\tau) ,$$

the last equality using the specified property of π.

Let k be a subring of $\mathbb{C}$ such that $k \cap \mathbb{Q} = \mathbb{Z}$. Let G be any group. Let $P \in \mathcal{P}(kG)$.

(4.4) Weak Conjecture. $r_P(1) = \sum_{\tau \in T(G)} r_P(\tau)$ $(= r_{P \otimes_{kG} k/k})$. Corollary (4.3) above affirms this when G is residually finite.

(4.5) Strong Conjecture. $r_P(s) = 0$ for all $s \neq 1$ in G, i.e. $r_P = r_{(kG)^n}$, where $n = r_{P \otimes_{kG} k/k}$.

We shall prove this below when G is a torsion free linear group (Corollary 6.4). It remains unproved for $G = SL_n(\mathbb{Z})$.

5. Characteristic p: Frobenius

Let k be a commutative ring of prime characteristic p and let A be a k-algebra. Then $T(A) = A/[A, A]$ is the commutator quotient of the restricted Lie algebra of A, so it inherits a p^{th} power map

$$F : T(a) \mapsto T(a)^p \overset{\text{def}}{=} T(a^p)$$

which is a Frobenius semi-linear endomorphism of T(A):

$$(t + t')^p = t^p + t'^p$$

$$(\alpha t)^p = \alpha^p t^p$$

for $t, t' \in T(A)$ and $\alpha \in k$. If M is an A-module of type (FP) we can apply the same considerations to $\mathrm{End}_A(M)$ to conclude that

$$T_M(u^p) = T_M(u)^p .$$

(One applies universality to the map $(M, u) \mapsto T_M(u^p)$, which is additive, linear, and commutative, to show that $T_M(u^p) = \phi(T_M(u))$ for some endomorphism ϕ of $T(A)$. Taking $M = A$ one sees that $\phi(t) = t^p$.)

Suppose now that A is a group algebra kG. Then

$$F^m\Big(\sum_{\tau \in T(G)} r(\tau)\tau\Big) = \sum_{\tau \in T(G)} r(\tau)^{p^m} \tau^{p^m}$$

$$= \sum_{\sigma \in T(G)} \Big(\sum_{\tau^{p^m} = \sigma} r(\tau)^{p^m} \Big)\sigma$$

where $T(s)^{p^m} = T(s^{p^m})$ for $s \in G$.

Remark. Suppose $s \in G$ and $T(s)^{p^m} = T(s)$, i.e. s is conjugate in G to s^{p^m}. If s has finite order n then n is prime to p clearly. Suppose that s has infinite order and say $s = ts^{p^m}t^{-1} = (tst^{-1})^{p^m}$. If $\alpha(x) = txt^{-1}$ for $x \in G$ then $\alpha^r(s)^{p^m} = \alpha^{r-1}(s)$, so $H = \langle s, \alpha(s), \ldots, \alpha^r(s), \ldots \rangle$ is a subgroup of G isomorphic to $\mathbb{Z}[\frac{1}{p}]$.

(5.1) Proposition. *Suppose that* $r = \sum r(\tau)\tau \in T(kG)$ *is fixed by* F^m. *Put* $S = \mathrm{supp}(r) = \{\tau \mid r(\tau) \neq 0\}$ *and let* R *denote the subring of* k *generated by all* $r(\tau)$.

(a) F^m *permutes* S *and* $r(\tau^{p^m}) = r(\tau)^{p^m}$ *for* $\tau \in S$. *Moreover* $r(1)^{p^m} = r(1)$.

(b) *If* $s \in G$ *and* $r(s) \neq 0$ *then* s *is conjugate in* G *to* $s^{p^{mn}}$ *for some* $n \leq |S|$.

(c) R *is a finite product of finite fields, and* $[R : \mathbb{F}_p] \leq p^{m \cdot |S|}$.

If $\sigma \in S$ then $r(\sigma) = \sum_{\tau^{p^m}=\sigma} r(\tau)^{p^m}$, so $\sigma = \tau^{p^m}$ for at least one $\tau \in S$. The consequent inclusion $S \subset F^m S$ of finite sets implies that F^m induces a bijection from S to S, whence (a) and (b). Let $n_1, \ldots, n_u$ denote the cardinals of the orbits in S of F^m; choose τ_i in the orbit of size n_i and put $r_i = r(\tau_i)$. Then $r_1, \ldots, r_u$ generate R and $r_i^{mn_i} = r_i$. Thus R is a quotient of the tensor product of the rings $\mathbb{F}_p[X]/(X^{p^{mn_i}} - X) \cong \prod_{d \mid mn_i} \mathbb{F}_{p^d}$, so R is semi-simple and of $\mathbb{F}_p$-dimension $\leq \prod_i p^{mn_i} = p^{m(n_1+\ldots+n_u)} = p^{m|S|}$.

(5.2) Corollary. If P is a kG-module of type (FP) then $r_P^p = r_P$ so the conclusions above apply, with $m = 1$, to $r = r_P$.

(5.3) Corollary. If k is an algebraically closed field and if $P \in \mathcal{P}(kG)$ then there is a finite field $k' \subset k$ and a $P' \in \mathcal{P}(k'G)$ such that $r_P = r_{P'}$.

It suffices to prove this instead for the algebraic closure k' of $\mathbb{F}_p$ in k. Then there is a finitely generated k'-algebra R in k so that P comes by base change from some $Q \in \mathcal{P}(RG)$, whence $r_P = r_Q$. Choose a retraction $\alpha : R \to k'$ and put $P' = Q \otimes_R k' \in \mathcal{P}(k'G)$. Then $r_{P'}(s) = \alpha r_Q(s) = r_Q(s)$ because, by Proposition (5.1), $r_Q(s) \in k'$ for all $s \in G$.

6. The complex group algebra

(6.1) Theorem. Let G be a group. Let $r = \sum_{\tau \in T(G)} r(\tau)\tau$ be the rank of a $\mathbb{C}G$-module of type (FP). Put $S = \mathrm{supp}(r)$, $r(G) = \{r(\tau) \mid \tau \in T(G)\}$, and $E = \mathbb{Q}(r(G))$, the subfield of $\mathbb{C}$ generated by $r(G)$.

(a) E is a finite abelian extension of $\mathbb{Q}$. Put $\Gamma = \mathrm{gal}(E/\mathbb{Q})$. There is a finite set Π of rational primes, including those that ramify in E, with the following properties.

(b) If $p \notin \Pi$ then $\tau \mapsto \tau^p$ is a permutation of S and for $\tau \in S$ we have $r(\tau^p) = \sigma r(\tau)$, where $\sigma = (p, E/\mathbb{Q})$, the Artin symbol.

Let $s \in G$ be such that $T(s) \in S$ (i.e. $r(s) \neq 0$).

(c) If $p \notin \Pi$ then s is conjugate in G to s^{p^n} for some $n \leq |S|$.

(d) Suppose that s has finite order m and put $w = e^{2\pi i/m}$. Then $s \in \mathbb{Q}(w)$, say $s = f(w)$ with $f(X) \in \mathbb{Q}[X]$. For all q prime to m we have $r(s^q) = f(w^q)$. In particular $r(1) \in \mathbb{Q}$ (Zalesskii [Z]).

(e) If s has infinite order then s belongs to a subgroup of G isomorphic to the semi-localization of $\mathbb{Z}$ at Π.

(f) If (every finitely generated subgroup of) G has a faithful linear representation over some field then $r(s) = 0$ whenever s has infinite order.

We can write $r = r_P - r_{P'}$ for some $P, P' \in \mathcal{P}(\mathbb{C}G)$. We claim there is a subring B of $\mathbb{C}$ with the following properties.

(1) B is finitely generated as a $\mathbb{Z}$-algebra.

(2) P, P' are isomorphic to $Q \otimes_B \mathbb{C}$ and $Q' \otimes_B \mathbb{C}$ respectively, for some $Q, Q' \in \mathcal{P}(BG)$.

(3) (i) $r(\tau) \neq 0 \Rightarrow r(\tau) \in B^\times$

(ii) $r(\tau) - r(\tau') \neq 0 \Rightarrow r(\tau) - r(\tau') \in B^\times$.

(4) B is integrally closed.

(5) If $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $B_1 = B \cap \overline{\mathbb{Q}}$, then the field of fractions of B_1 is a Galois extension of $\mathbb{Q}$ and B_1 is invariant under the Galois group.

Condition (2) is easily achieved with a finitely generated subring of $\mathbb{C}$; condition (3) requires only inversion of finitely many elements; then taking integral closure, to secure (4), preserves (1) because finitely generated $\mathbb{Z}$-algebras are excellent rings (cf. Matsumura [M]).

It suffices therefore to show that if we have B satisfying conditions (1), (2) and (3), then we can enlarge B to achieve (5) without jeopardizing (1). Let F denote the field of fractions of B and let $L' = F \cap \overline{\mathbb{Q}}$. Since F is a finitely generated extension of $\mathbb{Q}$, so also is L', which is therefore finite over $\mathbb{Q}$. Extend L' to a finite Galois extension L of $\mathbb{Q}$ and let $L \cdot F$ denote the compositum of L and F in $\mathbb{C}$. Since L' is algebraically closed in F, the field F is a regular extension of L', so $L \otimes_{L'} F$ is a field, hence isomorphic to $L \cdot F$. It follows that L is the algebraic closure of $\mathbb{Q}$ in $L \cdot F$, i.e. $L = L \cdot F \cap \overline{\mathbb{Q}}$.

Adjoining the ring of algebraic integers of L to B, we preserve (1) and reduce to the case where $L = L' = F \cap \overline{\mathbb{Q}}$ is Galois over $\mathbb{Q}$ and is the field of fractions of $B' = B \cap L$. By generic freeness we may invert some element of B' to make B/B' a free B'-module. Then if we enlarge B' to a localization B'_s which is invariant under $\mathrm{gal}(L/\mathbb{Q})$, the ring B_s will satisfy $B_s \cap \overline{\mathbb{Q}} = B'_s$ and so all the conditions of (5), as well as (1), and then its integral closure will satisfy (1)-(5).

Condition (1) implies that the ring $A = \mathbb{Z}[r(G)]$ is contained in B. To the rings $\mathbb{Z} \subset A \subset B$ we now apply the generic freeness lemma of Hochster-Roberts [H-R]:

(6) There is an integer $u \neq 0$ such that A_u and B_u/A_u are free modules over $\mathbb{Z}_u = \mathbb{Z}[\frac{1}{u}]$. Let p be a prime integer. For any $\mathbb{Z}$-module M put $\overline{M} = M/pM$. The natural map $\overline{M} \to \overline{M}_u = M_u/pM_u$ is bijective under the condition

(7) $p \nmid u$,

in which case we view it as an identification.

Assume (7). Since A_u is a direct summand of B_u (by (6)), the same is true of $\overline{A}_u$ in $\overline{B}_u$, so also of $\overline{A}$ in $\overline{B}$.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \overline{A} & \longrightarrow & \overline{B} \end{array} \qquad \begin{array}{c} Q \in \mathcal{P}(BG) \\ \downarrow \\ \overline{Q} \in \mathcal{P}(\overline{B}G) \end{array} \qquad \begin{array}{ccc} r_Q : G & \longrightarrow & B \\ \downarrow & & \downarrow \\ r_{\overline{Q}} : G & \longrightarrow & \overline{B} \end{array} \qquad \begin{array}{c} r \\ \downarrow \\ \overline{r} \end{array}$$

The composite $\overline{r} : G \to B \to \overline{B}$ is $r_{\overline{Q}} - r_{\overline{Q}'}$, where $\overline{Q}, \overline{Q}' \in \mathcal{P}(\overline{B}G)$, and it is fixed by Frobenius. Moreover $\overline{A} = \overline{\mathbb{Z}[r(G)]} = \mathbb{F}_p[\overline{r}(G)]$. Further condition (2)(i) implies that $\mathrm{supp}(\overline{r}) = \mathrm{supp}(r) = S$. It follows therefore from Proposition (5.1) that

(8) $\tau \mapsto \tau^p$ is a permutation of S

(9) $r(\tau^p) \equiv r(\tau)^p \mod pA$ for $\tau \in S$

(10) $\overline{A}$ is a finite product of finite fields.

These conditions are consequences of (7).

Since $\overline{A} = \overline{A}_u$ and A_u is a free $\mathbb{Z}_u$-module, we conclude that $[\overline{A} : \mathbb{F}_p] = [\overline{A}_u : \overline{\mathbb{Z}}_u] = [A_u : \mathbb{Z}_u] = [E : \mathbb{Q}]$, where $E = \mathbb{Q}(r(G))$ is the field of fractions of A. Since there exist primes p satisfying (7), we conclude that:

(11) E is a finite extension of $\mathbb{Q}$.

Let E' be the least Galois extension of $\mathbb{Q}$ containing E and put $\Gamma = \mathrm{gal}(E'/\mathbb{Q})$. It follows from condition (5) that

(12) E' is the field of fractions of $B' = B \cap E'$ and B' is Γ invariant.

Fix attention on some $\sigma \in \Gamma$. If $\mathfrak{Q}$ is a prime of E' unramified over $\mathbb{Q}$ and lying over the rational prime $p = p_{\mathfrak{Q}}$, then we have the Frobenius element $(\mathfrak{Q}, E'/\mathbb{Q}) \in \Gamma$. We have $\sigma = (\mathfrak{Q}, E'/\mathbb{Q})$ if and only if

(13) $\sigma(x) \equiv x^p \bmod \mathfrak{Q}$ for all $\mathfrak{Q}$-integral $x \in E'$.

Let Π_σ denote the set of primes $\mathfrak{Q}$ of E' unramified over $\mathbb{Q}$ such that $\sigma = (\mathfrak{Q}, E'/\mathbb{Q})$ and such that $p_{\mathfrak{Q}} \nmid u$ (condition (7)). It follows then from (5) and (6) that $B' = B \cap E'$ consists of $\mathfrak{Q}$-integral elements, so we may identify each $\mathfrak{Q} \in \Pi_\sigma$ with a maximal ideal of B'. It follows from the Čebotarev Density Theorem that

(14) Π_σ is infinite.

Let $\tau \in S$. Let $\mathfrak{Q} \in \Pi_\sigma$ and $p = p_{\mathfrak{Q}}$. Then since $r(\tau) \in A \subset B'$ it follows from (13) that

(15) $\sigma r(\tau) \equiv r(\tau)^p \bmod \mathfrak{Q}$.

Combining (15) with (9) (which is available because $p \nmid u$, by definition of Π_σ) we obtain

(16) $\sigma r(\tau) \equiv r(\tau^p) \bmod \mathfrak{Q}$.

(We use the obvious fact that $pA \subset \mathfrak{Q}$.) Now as $\mathfrak{Q}$ varies over the infinite set Π_σ, we see infinitely many $p = p_{\mathfrak{Q}}$ as well, whereas, by (8), τ^p varies over the finite set S. Therefore there is a $\tau_1 \in S$ such that $\tau_1 = \tau^p$, where $p = p_{\mathfrak{Q}}$, for infinitely many primes $\mathfrak{Q} \in \Pi_\sigma$. Then (16) implies that $\sigma r(\tau) - r(\tau_1)$ belongs to infinitely many primes $\mathfrak{Q} \in \Pi_\sigma$, whence $\sigma r(\tau) = r(\tau_1)$. If we now vary $\tau \in S$ and $\sigma \in \Gamma$ we conclude that:

(17) Γ permutes $r(G)$, hence $E = \mathbb{Q}(r(G))$ is a Galois extension of $\mathbb{Q}$ (i.e. $E = E'$).

Returning now to the discussion above, but now armed with (17), we conclude from (2)(ii) that the congruence (16) is even an equality:

(18) $\sigma r(\tau) = r(\tau^p)$ if $\mathfrak{Q} \in \Pi_\sigma$ and $p = p_{\mathfrak{Q}}$.

Combining (18) with (9) we obtain the congruence

(19) $\sigma r(\tau) \equiv r(\tau)^p \bmod pA$, if $\mathfrak{Q} \in \Pi_\sigma$ and $p = p_{\mathfrak{Q}}$ whence

(19') $\sigma(x) \equiv x^p \bmod pA$, if $x \in A$, $\mathfrak{Q} \in \Pi_\sigma$ and $p = p_{\mathfrak{Q}}$.

If we avoid the finite set of primes p which figure in the conductor of the integral closure of A over A, then (19') implies that $\sigma = (\mathfrak{Q}, E/\mathbb{Q})$, whereas it is a condition depending only on $p = p_{\mathfrak{Q}}$; whence $\sigma = (\mathfrak{Q}', E/\mathbb{Q})$ for every prime $\mathfrak{Q}'$ lying over p. But varying $\mathfrak{Q}'$ over the primes above a given p varies $(\mathfrak{Q}', E/\mathbb{Q})$ over a conjugacy class of Γ. It follows therefore that σ coincides with its conjugates in Γ. Since σ was an arbitrary element of Γ, we conclude the following:

Let Π denote the set of rational primes which divide u or ramify in E.

(20) Γ is abelian. If p is a prime not in Π then, for $\tau \in S$, $r(\tau^p) = \sigma r(\tau)$ where $\sigma = (p, E/\mathbb{Q}) \in \Gamma$ is the Artin symbol. By the theorem of Kronecker-Weber, E is contained in a cyclotomic field.

Let $s \in G$ and suppose that $r(s) \neq 0$.

Suppose first that s has finite order m. Let p be a prime not dividing u and unramified in E, and put $\sigma = (p, E/\mathbb{Q})$. Then $\sigma r(s) = r(s^p)$, so σ fixes $r(s)$ if $p \equiv 1 \bmod m$. It follows that $r(s) \in \mathbb{Q}(w)$ where $w = e^{2\pi i/m}$. Writing $r(s) = f(w)$ with $f(X) \in \mathbb{Q}[X]$, we have $\sigma r(s) = f(w^p)$, whence $r(s^p) = f(w^p)$. It follows from this that $r(s^q) = f(w^q)$ for all integers q prime to m.

Suppose that s has infinite order. From (8) we see that for any prime $p \notin \Pi$ there is an integer $n \leq |S|$ such that s is conjugate in G to s^{p^n}, hence to s^{p^N} where $N = |S|!$ It follows that s is conjugate in G to s^{q^N} for any integer q not divisible by any primes in Π. It is then easy to embed s in a subgroup of G isomorphic to the additive subgroup of $\mathbb{Q}$ consisting of elements with denominators not divisible by primes in Π.

Note finally that the modules P, $P' \in \mathcal{P}(\mathbb{C}G)$ in the theorem arise by base change from modules V, $V' \in \mathcal{P}(\mathbb{C}H)$ where H is some finitely generated subgroup of G, and so $r = r_P - r_{P'}$ is the image of $r_1 = r_V - r_{V'}$ under $T(\mathbb{C}H) \to T(\mathbb{C}G)$. Then $S = \mathrm{supp}(r)$ consists of conjugacy classes in G which meet some conjugacy class of H in $\mathrm{supp}(r_1)$. If r_1 vanishes on elements of infinite order, the same is therefore true of r.

Combining the observations above we conclude that

(6.2) Proposition. $r(s) = 0$ whenever s has infinite order, provided that G satisfies the following condition:

(D) An element $s \in G$ has finite order if, for some finitely generated subgroup H containing s, and some integer $N \geq 1$, s is conjugate in H to s^{p^N} for all but finitely many primes p.

Of course (D) holds if the following condition (D') holds.

(D') An element $s \in G$ has finite order if, in some finitely generated subgroup H containing s, s is a p^{th} power for infinitely many primes p.

(6.3) Theorem. Linear groups satisfy (D'), hence also (D).

(6.4) Corollary. Let G be a torsion free linear group. Let $P \in \mathcal{P}(\mathbb{C}G)$. Then $r_P(s) = 0$ for all $s \neq 1$.

Corollary (6.4) establishes the Strong Conjecture (4.5) for torsion free linear groups, even with $\mathbb{C}$ in place of $\mathbb{Z}$. However it is not clear how to handle linear groups with torsion, even groups like $G = SL_n(\mathbb{Z})$. Here is an approach that makes some reduction in the problem.

Let $P \in \mathcal{P}(\mathbb{Z}G)$, $r = r_P$, and $s \in G$ be such that $r(s) \neq 0$. We know from Proposition (6.2) and Theorem (6.3) that s has finite order, and the Strong Conjecture asserts that $s = 1$. Choose a normal torsion free subgroup N of finite index in G, for example a principal congruence subgroup. Then, as in the proof of Theorem (4.1), we can replace G by the semi-direct product $H = N \cdot \langle s \rangle$. Consider $S = \mathrm{supp}(r) \subset T(H)$. By shrinking N to a smaller subgroup N_1, we can try to make as many elements of S (other than $T_H(s)$) as possible escape from $H_1 = N_1 \cdot \langle s \rangle$. Then choosing another normal subgroup $N_2 \subset N_1$ of finite index and 'sufficiently small', we can hope to distinguish $T_{H_1}(s)$ from the other elements of $\mathrm{supp}(r)$ in $H_1/N_2 = (N_1/N_2) \cdot \langle s \rangle$. If this is accomplished we could conclude from Theorem (4.1) applied to the latter finite group, that $r(s) = 0$ if $s \neq 1$.

Theorem (6.3) asserts explicitly: Let F be a field and let G be a finitely generated subgroup of $GL_n(F)$. Let $s \in G$ and suppose that, for infinitely many primes p, there is an $x \in G$ such that $x^p = s$. Then s

has finite order.

To prove this note first that there is a finitely generated subring A of F such that $G \subset GL_n(A)$. It suffices to prove the above assertion with G replaced by $GL_n(A)$, so suppose that $G = GL_n(A)$. After enlarging A slightly (i. e. by a finite integral extension) we may further assume that:

(i) the set spec(s) of eigenvalues of s is contained in A; and

(ii) A is integrally closed in its field of fractions, which we may take to be F.

Let $\overline{F}$ be an algebraic closure of F. The following lemma uses only that F is a finitely generated extension of its prime field.

(6.5) Lemma. ([B], Propositions (A.3) and (A.4).) *There is an integer $N > 0$ such that if $\alpha \in \overline{F}$ and $[F(\alpha) : F] \leq n$ then:*

(a) $\alpha^N = 1$ *if α is a root of unity;*

(b) $\alpha \in F$ *if $\alpha^m \in F$ for some m prime to N.*

Let $a \in \text{spec}(s) \subset A$. If $s = x^p$ with p prime and $x \in GL_n(A)$ then $a = \alpha^p$ for some $\alpha \in \text{spec}(x)$ and $[F(\alpha) : F] \leq n$ (α being a root of the characteristic polynomial of x). If $p \nmid N$ (N as in the lemma) we then have $\alpha \in F$, whence $\alpha \in A^\times$ since α and α^{-1} are integral over A. Thus $a \in A^\times$ is a p^{th} power in $A^\times$ for infinitely many primes p. But $A^\times$ is a finitely generated abelian group ([L], II, §4, Corollary of Theorem 5), so a must have finite order. It follows that some power $u = s^q$ of s has all eigenvalues equal to 1. If $s = x^p$ then $u = (x^q)^p$, so u inherits the hypothesis made on s.

It suffices to show that the unipotent u has finite order. Suppose p is prime and $u = x^p$ with $x \in GL_n(A)$. Then the eigenvalues of x are p^{th} roots of unity of degree $\leq n$ over F. Lemma (6.5) (part (a)) thus implies that x is unipotent if $p \nmid N$. If $\text{char}(F) > 0$, then unipotents have finite order and we are done. So assume that $\text{char}(F) = 0$. Then exp and log give inverse bijections between the unipotent elements in $GL_n(F)$ and the nilpotent elements in $M_n(F)$. If x is unipotent and $x^p = u$ we therefore have $x = \exp(\frac{1}{p}\log(u))$. Let $u = 1 + v$. $L = \log(u) = v - \frac{v^2}{2} + \frac{v^3}{3} - \ldots + (-1)^{n-1}\frac{v^n}{n}$ and $E(t) = \exp(tL) = 1 + tL + \frac{t^2L^2}{2!} + \ldots + \frac{t^nL^n}{n!}$

This is a polynomial in t with matrix coefficients in $M_n(B)$ where $B = A[\frac{1}{n!}]$. We have $u = E(1)$ and we know that $E(\frac{1}{p}) \in GL_n(A)$ for infinitely many primes p. The next lemma shows that $E(t)$ is a constant, so $u = E(1) = E(0) = 1$.

(6.6) Lemma. *Let B be a finitely generated subring of a field F of characteristic 0. Let $f(t) = b_0 + b_2 t + \ldots + b_n t^n$, $b_n \neq 0$, be a polynomial in $B[t]$ such that $f(\frac{1}{p}) \in B$ for infinitely many primes p. Then f is a constant.*

Let C be a finitely generated integrally closed subring of F containing B and b_i^{-1} for all $b_i \neq 0$. All but finitely many primes p are not invertible in C. Choose one such that $f(\frac{1}{p}) \in C$ and let v be a discrete valuation of C such that $v(p) > 0$. Then

$$0 \leq v(f(\tfrac{1}{p})) = v(b_0 + \frac{b_1}{p} + \ldots + \frac{b_n}{p^n}) = -nv(p)$$

because $v(b_i) = 0$ for all $b_i \neq 0$, and $b_n \neq 0$.

We close this section with a very pretty application of the above and other K-theoretic results.

(6.7) Theorem (Farkas-Snider [F-S]). *Let G be a virtually polycyclic group. If G is torsion free then $\mathbb{C}G$ has no zero divisors.*

Let H be a polyinfinite cyclic normal subgroup of finite index in G. Then $A = \mathbb{C}H$ is obtained from $\mathbb{C}$ by a finite succession of twisted Laurent polynomial extensions, so it has the following properties (see [F-H]).

(1) A is a (left and right) noetherian ring without zero divisors.

(2) A has finite global dimension.

(3) $K_0(A)$ is infinite cyclic, generated by $[A]$.

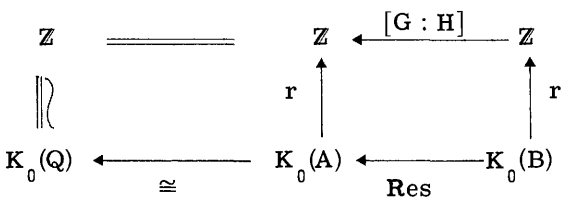
The ring $B = \mathbb{C}G$ is a finitely generated (left and right) A-module. In fact

(4) B is a (left and right) noetherian prime ring (see [P]).

Since G has finite virtual cohomological dimension and G is torsion free it follows from a theorem of Serre [S] that G has finite cohomological dimension, whence

(5) B has finite global dimension.

Since G and H clearly satisfy condition (D), and are torsion free, we can, by Proposition (6.2), identify the rank with a homomorphism from K_0 of the group algebra to $\mathbb{Z}$, taking the value 1 on the class of the ring itself. Now we have a commutative diagram (since $r_{P/H}(1) = [G : H]r_{P/G}(1)$ for $P \in \mathcal{P}(\mathbb{C}G)$)



where Q is the division ring of fractions of A. It follows that if M is any finitely generated B-module, hence automatically of type (FP), then $M \otimes_A Q$ is a Q-vector space of dimension divisible by $[G : H] = \dim_Q(B \otimes_A Q)$. This implies that B cannot contain a direct sum of two non zero submodules. From (4) it follows that B has a full ring of fractions of the form $F = M_n(D)$, where D is a division ring. If $n \geq 2$ there is an idempotent $e \neq 0$ or 1 in F. Then $(B \cap Fe) \oplus (B \cap F(1-e)) \subset B$ violates the conclusion drawn above, whence the theorem.

7. Euler characteristics

We summarise here some results from [B], §10.

(7.1) Notation. Let k be a commutative ring and let G be a group Let $r : T(G) \to k$ be a function with finite support. As usual we write $r(s) = r(T(s))$ for $s \in G$. Define $\bar{r} : T(G) \to k$ and $\sum(r) \in k$ by:

(1) $\bar{r}(s) = r(s^{-1})$, $\sum(r) = \sum_{\tau \in T(G)} r(\tau)$.

If $M \in \mathcal{R}_k(kG)$ has character χ_M then $\bar{\chi}_M = \chi_{M^v}$, where $M^v = \mathrm{Hom}(M, k)$ is the contragredient module. If $P \in \mathcal{P}(kG)$ then

(2) $M \otimes_k P$, $\mathrm{Hom}_k(M, P) \in \mathcal{P}(kG)$ and

(3) $r_{M \otimes_k P} = \bar{\chi}_M \cdot r_P$,

(4) $r_{\mathrm{Hom}_k(M, P)} = \chi_M \cdot r_P$.

For $r \in T(kG)$ as above $\sum(r)$ is the image of r' under the map $T(kG) \to T(k) = k$ induced by the augmentation $kG \to k$. In particular

(5) $\sum(r_P) = r_{P \otimes_{kG} k/k}$.

(7.2) Euler characteristics. We say that G is of type (FP) over k if k is a kG-module of type (FP), i.e. if there is an exact sequence

(6) $0 \to P_n \to \dots \to P_0 \to k \to 0$

with each $P_i \in \mathcal{P}(kG)$. We then call

(7) $\chi_G = r_{k/kG} = \sum_i (-1)^i r_{P_i} : G \to k$

the complete Euler characteristic of G over k. Chiswell [Ch] and the author introduced the Euler characteristic

(8) $\chi(G) = \chi_G(1) \in k$,

and Ken Brown introduced a homological Euler characteristic which, in the present case, can be described as

(9) $\sum(\chi_G) = \sum_{\tau \in T(G)} \chi_G(\tau)$.

If $\phi : k \to k'$ is a homomorphism of commutative rings then G is of type (FP) over k' and $\phi \circ \chi_G$ is its complete Euler characteristic over k'.

(7.3) Finite groups. A finite group G is of type (FP) over k if and only if $|G| \in k^\times$. In this case Proposition (3.1) implies that

(10) $\chi_G(s) = 1/|Z_G(s)|$ for all $s \in G$. For $s = 1$ this yields $\chi(G) = 1/|G|$. Further $\sum(\chi_G) = 1$.

(7.4) Homology. Suppose that G is of type (FP) over k. Let $M \in \mathcal{R}_k(kG)$. Then the sequence $M \otimes_k (6)$ of kG-modules is exact, showing (2) that M is of type (FP) over k and (3) that

(11) $r_{M/kG} = \overline{\chi}_M \cdot \chi_G$.

Suppose that, for all i, $H_i(G, M)$ (respectively, $H^i(G, M)$) is of type (FP) over k; denote its k-rank $h_i(M)$ (respectively, $h^i(M)$). Then

(12) $\sum(r_{M/kG}) = \sum_i (-1)^i h_i(M)$, respectively

(13) $\sum(r_{M^v/kG}) = \sum_i (-1)^i h^i(M)$.

When $M = k$ we write these formulae as

(14) $\sum(\chi_G) = \sum_i (-1)^i h_i = \sum_i (-1)^i h^i$.

This formula identifies $\sum(\chi_G)$ with the Euler characteristic of Ken Brown and explains the terminology 'homological Euler characteristic'.

(7.5) Finite index. Let H be a subgroup of finite index in G. Suppose that G is of type (FP) over k. Then H is likewise, and

(15) $\chi_H(s) = \chi_G(s) \cdot [Z_G(s) : Z_H(s)]$ for $s \in H$.

When $s = 1$:

(16) $\chi(H) = \chi(G) \cdot [G : H]$.

Ken Brown has shown, when $k = \mathbb{Z}$, that

(17) $\sum(\chi_H) = \sum(\chi_G) \cdot [G : H]$,

though this may fail for general k.

(7.6) Conjectures. Let G be of type (FP) over k.

Case $k = \mathbb{Z}$.

(18) Strong Conjecture (4.5) $\Rightarrow$ $\chi_G(s) = 0$ for all $s \neq 1$ in G.

(19) Weak Conjecture (4.4) $\Rightarrow$ $\chi(G) = \sum(\chi_G)$.

Case $k = \mathbb{Q}$. We conjecture that:

(20) $\chi_G(s) = 0$ if s has infinite order.

(21) G has only finitely many conjugacy classes of elements of finite order. Serre conjectures that

(22) $\chi_G(s) = \chi(Z_G(s))$ if s has finite order, assuming that $Z_G(s)$ is of type (FP) over k. This has been proved in many interesting cases by Ken Brown. For finite G it is affirmed in (7.3) above.

(7.7) Normal subgroups of type (FP). Let $\varepsilon : H \hookrightarrow G$ be the inclusion of a normal subgroup of type (FP) and let $\pi : G \to G' = G/H$ be the natural projection. We have (using (1.6))

(23) $\chi_H(t) = \chi_H(\alpha t)$ for all $\alpha \in \mathrm{Aut}(H)$, $t \in H$,

so that the finite set $\mathrm{supp}(\chi_H) \subset T(H)$ is $\mathrm{Aut}(H)$-invariant. From (23) it follows that

(24) $\chi_H(sts^{-1}) = \chi_H(t)$ for all $s \in G$, $t \in H$.

For $t \in H$ put

(25) $n_t = [G : H \cdot Z_G(t)]$ = the number of H-conjugacy classes con-

tained in $T_G(t)$. Then:

(26) $n_t = \infty \Rightarrow \chi_H(t) = 0$ and $\varepsilon_*\chi_H(t) = 0$,

(27) $n_t < \infty \Rightarrow \varepsilon_*\chi_H(t) = n_t \cdot \chi_H(t)$.

Thus (when $t = 1$)

(28) $\varepsilon_*\chi_H(1) = \chi(H)$ and

(29) $\sum(\varepsilon_*\chi_H) = \sum(\chi_H)$.

Since k is a kH-module of type (FP) it follows, since kG is a free kH-module, that $kG' = k \otimes_{kH} kG$ is a kG-module of type (FP). Consequently (see (1.7)) $\pi : kG \to kG'$ induces a k-linear map $\pi^* : T(kG') \to T(kG)$. Moreover

(30) $\pi^* T_{G'}(1) = \varepsilon_* \chi_H$.

(7.8) Theorem. ([B] and [St2].) Let $\tau' \in T(G')$. Then $\pi^*(\tau')$ is a linear combination of classes $\tau \in T(G)$ such that $\pi(\tau) = \tau'$. Hence there is an element $L(\tau') \in k$ such that

(31) $\pi_*\pi^*(\tau') = L(\tau')\tau'$ and

(32) $L(1) = \sum(\chi_H)$.

If the k-modules $H_i(H, k)$ are of type (FP) (e.g. if k is a field or a P.I.D.) then $\overline{L} : s' \mapsto L(s'^{-1})$ is the virtual character of the natural action (via conjugation) of G' on $H_*(H, k)$.

(7.9) Suppose that G' acts trivially on $H_*(H, k)$. This happens, for example, if $G = H \cdot Z_G(H)$. Then $L(s') = L(1) = \sum(\chi_H)$ for all $s' \in G'$, so

(33) $\pi_*\pi^*(r') = \sum(\chi_H) \cdot r'$ for all $r' \in T(kG')$.

(7.10) Suppose that G' is of type (FP) over k. Then G is likewise, and

(34) $$\chi_G = \pi^*(\chi_{G'}) = \sum_{\tau' \in T(G')} \chi_{G'}(\tau')\pi^*(\tau').$$

If $s \in G$ and $\pi s = s' \in \tau' \in T(G')$ then it follows from (34) and (7.8) that

(35) $\chi_G(s) = \chi_{G'}(s') \cdot (\pi^*\tau')(s)$.

When $s \in H$ it follows from (35) and (30) that

(36) $$\chi_G(s) = \begin{cases} 0 & \text{if } n_s = \infty \\ \chi(G') \cdot \chi_H(s) \cdot n_s & \text{otherwise} \end{cases}$$

where n_s is as in (25). For $s = 1$ this yields the extension formula

(37) $\chi(G) = \chi(G')\chi(H)$.

From (31) and (34) we further conclude that

(38) $\pi_*(\chi_G) = L \cdot \chi_{G'}$.

Explicitly, for $\tau' \in T(G')$,

(39) $\sum_{\pi\tau=\tau'} \chi_G(\tau) = L(\tau')\chi_{G'}(\tau')$.

Hence, for $\tau' = 1$, we obtain, using (32),

(40) $\sum_{\substack{\tau\in T(G) \\ \tau\subset H}} \chi_G(\tau) = \Sigma(\chi_H) \cdot \chi(G')$ and

(41) $\Sigma(\chi_G) = \sum_{\tau'\in T(G')} L(\tau')\chi_{G'}(\tau')$.

When, further, G' acts trivially on $H_*(H, k)$ as in (7.9), then (38) becomes

(42) $\pi_*(\chi_G) = \Sigma(\chi_H) \cdot \chi_{G'}$, whence

(43) $\Sigma(\chi_G) = \Sigma(\chi_H) \cdot \Sigma(\chi_{G'})$.

(7.11) Suppose that $\chi_{G'}(s') = 0$ for all $s' \neq 1$ in G' (e.g. that G' satisfies the Strong Conjecture (4.5) over k). Then (34) becomes, in view of (30) and (36),

(44) $\chi_G = \chi(G') \cdot \varepsilon_*\chi_H$.

Explicitly, for $s \in H$,

(45) $\chi_G(s) = \chi(G') \cdot \chi_H(s)n_s$ with n_s as in (25),

and both sides of (45) vanish if $n_s = \infty$. For $s \notin H$, $\chi_G(s) = 0$. Further (43) is valid in this case. Applying π_* to (44) we conclude from (38) that $\pi_*(\chi_G) = \chi(G')\chi(H) \cdot T_{G'}(1)$.

(7.12) Suppose that G' is finite and that k is an integral domain of characteristic zero in which no prime divisor of $|G'|$ is invertible. Suppose that G is of type (FP) over k. Then

(46) $\pi_*(\chi_G) = (\Sigma(\chi_H)/|G'|) \cdot T_{G'}(1)$ and

(47) $L = \Sigma(\chi_H) \cdot \rho$

where $\rho = \chi_{kG'}$ is the character of the regular representation of G' over k.

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2 · Groups of virtually finite dimension

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The purpose of this paper is to give an exposition of two topics in the theory of groups of finite virtual cohomological dimension: (a) the theory of Euler characteristics and (b) the recently developed Farrell cohomology theory. These are treated in Parts II and III, respectively. Part I is devoted to a review of the necessary background material.

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PART I. REVIEW

Good references for the material of Part I are [5] and [30].

§1. Finiteness conditions

Recall that the homology and cohomology of a group Γ can be define algebraically, in terms of projective resolutions, as follows. Regard $\mathbb{Z}$ as a module (with trivial Γ-action) over the integral group ring $\mathbb{Z}\Gamma$, and choose a projective resolution $P = (P_i)_{i\geq 0}$:

$$\ldots \to P_1 \to P_0 \to \mathbb{Z} \to 0 .$$

One then defines, for any Γ-module M,

$$H_*(\Gamma, M) = H_*(P \otimes_{\mathbb{Z}\Gamma} M) \quad \text{and} \quad H^*(\Gamma, M) = H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(P, M)).$$

[Note: We have been sloppy here about the distinction between left modules and right modules. To avoid ambiguity, let us agree that all modules in this paper are to be understood as left modules unless the contrary is

explicitly stated. But then in order to make sense out of the tensor product above, one must convert P to a complex of right modules in the usual way, by setting $x\gamma = \gamma^{-1}x$ for $x \in P_i$, $\gamma \in \Gamma$.]

Alternatively, the homology and cohomology groups $H(\Gamma, M)$ can be defined topologically, in terms of Eilenberg-MacLane complexes. One chooses an Eilenberg-MacLane complex of type $K(\Gamma, 1)$, i.e., a connected CW-complex Y such that $\pi_1 Y = \Gamma$ and $\pi_i Y = 0$ for $i > 1$, and one sets

$$H(\Gamma, M) = H(Y, M),$$

where the groups on the right are to be interpreted as homology and cohomology groups with local coefficients. [The equivalence of the algebraic and topological definitions follows from the fact that the universal cover $\tilde{Y}$ of Y is contractible, so that its chain complex $C(\tilde{Y})$ provides a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.]

(1.1) Example. Suppose Γ is a discrete subgroup of a Lie group G which has only finitely many connected components. Let K be a maximal compact subgroup of G and let X be the homogeneous space G/K. One knows that X is diffeomorphic to Euclidean space $\mathbb{R}^d$ ($d = \dim G - \dim K$) and that Γ acts properly on X (i.e., every point $x \in X$ has a neighbourhood U such that $\gamma U \cap U \neq \emptyset$ for only finitely many $\gamma \in \Gamma$). In particular, every isotropy group Γ_x is finite. If we now assume that Γ is torsion-free, then these isotropy groups are trivial, so that Γ acts freely on X and the projection $X \to \Gamma\backslash X$ is a covering map. Since X is contractible, it follows that the manifold $\Gamma\backslash X$ is a $K(\Gamma, 1)$, hence

$$H(\Gamma, M) \approx H(\Gamma\backslash X, M).$$

In the definitions of $H(\Gamma, M)$ above, one is free to choose the resolution P or the Eilenberg-MacLane complex Y. It is therefore natural to try to take them to be as 'small' as possible, and this leads to various finiteness notions. For example, if we interpret 'small' in terms of dimension, then we arrive at the notion of cohomological dimension:

one says that Γ has finite cohomological dimension if the following conditions, which are known to be equivalent, are satisfied:

(i) $\mathbb{Z}$ admits a projective resolution over $\mathbb{Z}\Gamma$ of finite length, i. e. a resolution of the form

$$0 \to P_n \to \dots \to P_0 \to \mathbb{Z} \to 0.$$

(Such a resolution is said to have length $\leq n$.)

(ii) $\mathbb{Z}$ admits a free resolution over $\mathbb{Z}\Gamma$ of finite length.

(iii) There is an integer n such that $H^i(\Gamma, M) = 0$ for $i > n$ and all Γ-modules M.

(iv) There exists a finite dimensional $K(\Gamma, 1)$-complex.

If these conditions are satisfied then we define the cohomological dimension of Γ (denoted cd Γ) to be the minimal length of a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$; otherwise we set cd $\Gamma = \infty$. It is known that cd Γ is also equal to the minimal length of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, as well as to the smallest integer n satisfying (iii). If cd $\Gamma \neq 2$ then cd Γ can also be described topologically, as the minimal dimension of a $K(\Gamma, 1)$-complex, but it is not known whether this is true if cd $\Gamma = 2$; in this case one knows only that there exists a $K(\Gamma, 1)$ of dimension ≤ 3.

The torsion-free discrete subgroups of Lie groups as in 1.1 provide examples of groups of finite cohomological dimension. On the other hand, any group with torsion has infinite cohomological dimension. (In case Γ is a non-trivial finite cyclic group, this is proved by a direct calculation of $H^*(\Gamma)$, which is non-trivial in arbitrarily high dimensions; the general case follows from the elementary fact that cd $\Gamma' \leq$ cd Γ whenever $\Gamma' \subset \Gamma$.

Further finiteness conditions are obtained by requiring not only that the projective resolution P be of finite length, but also that each module P_i be finitely generated. Such a resolution is said to be finite, and we will say that Γ is of type (FP) (resp. (FL)) if $\mathbb{Z}$ admits a finite projective (resp. free) resolution over $\mathbb{Z}\Gamma$. Unlike the situation in the definition of cd Γ above, where we allowed infinitely generated modules, there is no reason to expect that a group of type (FP) is necessarily of type (FL). Nevertheless, the surprising fact is that there are no known examples of groups of type (FP) which are not of type (FL). Indeed,

such a group would necessarily have a non-trivial projective class group $\tilde{K}_0(\mathbb{Z}\Gamma)$, and there are no known examples of torsion-free groups Γ such that $\tilde{K}_0(\mathbb{Z}\Gamma) \neq 0$. In spite of this lack of examples, however, we will see below (cf. 5.4) that the theory of groups of type (FP) has concrete applications.

The (FP) and (FL) conditions have reasonable topological interpretations, at least if we assume that Γ is finitely presented: If Γ is finitely presented then Γ is of type (FL) if and only if there exists a $K(\Gamma, 1)$ which is a finite CW-complex, and Γ is of type (FP) if and only if some (and hence every) $K(\Gamma, 1)$ is finitely dominated, i.e. is a retract, in the homotopy category, of a finite complex. [Note: It is not known whether the (FP) (or (FL)) condition implies that Γ is finitely presented; if so, then the finite presentation assumption above can be dropped.]

For example, if Γ is a torsion-free subgroup of a Lie group G as in 1.1, and if Γ is co-compact (i.e. G/Γ is compact), then Γ is of type (FL). More interestingly, all torsion-free arithmetic groups are of type (FL) even though they are rarely co-compact (see Serre's lectures [32]).

We close this section by discussing the behavior of the finiteness conditions with respect to passage to subgroups of finite index.

(1.2) Theorem (Serre [30]). Let Γ be a torsion-free group and Γ' a subgroup of finite index. Then $\operatorname{cd}\Gamma' = \operatorname{cd}\Gamma$.

(1.3) Corollary. If Γ and Γ' are as in 1.2, then Γ is of type (FP) if and only if Γ' is of type (FP).

Remark. It is not known whether the analogous statement for groups of type (FL) is true.

We will now sketch the proof of the theorem; the corollary is left as an exercise for the reader. Assuming first that $\operatorname{cd}\Gamma < \infty$, it is easy to prove $\operatorname{cd}\Gamma' = \operatorname{cd}\Gamma$. For if $\operatorname{cd}\Gamma = n$ then the functor $H^n(\Gamma, -)$ is right exact, hence $H^n(\Gamma, F) \neq 0$ for some free $\mathbb{Z}\Gamma$-module F; letting F' be the free $\mathbb{Z}\Gamma'$-module of the same rank, we have $H^n(\Gamma', F') \approx H^n(\Gamma, F)$ (this is a special case of 'Shapiro's lemma'), so $\operatorname{cd}\Gamma' \geq n = \operatorname{cd}\Gamma$. The

opposite inequality is trivial.

It remains to prove that if $\operatorname{cd} \Gamma' < \infty$ then $\operatorname{cd} \Gamma < \infty$. Let X' be a finite-dimensional contractible simplicial complex on which Γ' acts freely (and simplicially), i. e. X' is the universal cover of a finite-dimensional simplicial $K(\Gamma', 1)$. By a 'multiplicative induction' construction (see [30], 1. 7, or [25], II, §16) one produces a simplicial Γ-complex X whose underlying simplicial complex is isomorphic to the product of $(\Gamma : \Gamma')$ copies of X'; in particular, X is contractible and finite dimensional. Moreover, the action of Γ on X is proper. Using now the hypothesis that Γ is torsion-free, we see that Γ acts freely on X, so that X/Γ is a finite dimensional $K(\Gamma, 1)$ and $\operatorname{cd} \Gamma < \infty$.

§2. Virtual notions

Groups Γ with torsion, as we have seen, cannot satisfy any of the finiteness conditions of §1. There will, however, often be torsion-free subgroups Γ' of finite index which do satisfy the finiteness conditions. (For example, in the arithmetic case we have the congruence subgroups.) We are thus led to introduce 'virtual' finiteness conditions.

Let Γ be a group which is virtually torsion-free, i. e. which has a torsion-free subgroup of finite index. By Serre's theorem (1. 2), all such subgroups have the same cohomological dimension, and this common dimension is called the virtual cohomological dimension of Γ, denoted $\operatorname{vcd} \Gamma$. Similarly, we say that Γ is of type (VFP) (resp. (VFL)) if Γ has a subgroup of finite index of type (FP) (resp. (FL)). If Γ is of type (VFP) then Corollary 1. 3 implies that every torsion-free subgroup of finite index is of type (FP). The analogous statement for groups of type (VFL) is not known, and one therefore introduces the following apparent strengthening of the (VFL) condition: A virtually torsion-free group is said to be of type (WFL) if every torsion-free subgroup of finite index is of type (FL). The main examples of groups of type (WFL) are the arithmetic groups, as well as the S-arithmetic groups in the reductive case (cf. [8], [9], [32]).

This paper is concerned with groups Γ such that $\operatorname{vcd} \Gamma < \infty$. This condition has the following topological interpretation, which follows

immediately from the proof of Theorem 1.2:

(2.1) Proposition. Let Γ be a virtually torsion-free group. Then $\operatorname{vcd}\Gamma < \infty$ if and only if there exists a finite-dimensional contractible simplicial complex X on which Γ acts properly (and simplicially).

One should think of X as an analogue of the homogeneous space G/K which is available if Γ is a discrete subgroup of a Lie group, cf. 1.1.

For future reference we record the following fact, which comes from an examination of Serre's construction used in the proof of Theorem 1.2:

(2.2) Addendum. If $\operatorname{vcd}\Gamma < \infty$ then the space X in 2.1 can be chosen so that the fixed-point set X^H is contractible for every finite subgroup $H \subseteq \Gamma$.

Questions. 1. Can X always be chosen so that $\dim X = \operatorname{vcd}\Gamma$? We will see in the examples below a number of cases where this is known to be true, but the general case remains open, even if Γ is arithmetic. Note, in particular, that if Γ has torsion then the space X constructed by Serre in the proof of Theorem 1.2 always has $\dim X \geq 2 \cdot \operatorname{vcd}\Gamma$, except in the trivial case where Γ is finite and X is a point.

2. What algebraic finiteness conditions on Γ will guarantee that X can be chosen so that X/Γ is compact? For arithmetic groups such an X exists by Borel-Serre [8] and the equivariant triangulation theorem [21]. Even for S-arithmetic groups, however, the question seems to be open, the problem being the existence of an equivariant triangulation (cf. [9, p231]). Note, again, that Serre's construction in the proof of 1.2 will never produce an X with compact quotient, unless Γ is finite.

Examples. 1. $\operatorname{vcd}\Gamma = 0$ if and only if Γ is finite.

2. $\operatorname{vcd}\Gamma \leq 1$ if and only if Γ is the fundamental group of a graph of finite groups of bounded order. This result is a generalization of the theorem of Stallings [35] and Swan [40] that groups of cohomological dimension 1 are free. See [28] for a proof and further references; see

also [31], ch. II, 2.6. (Note, in this case, that one does have a contractible 1-dimensional complex on which Γ operates properly.)

3. If Γ is a (finitely generated) 1-relator group then Γ is of type (WFL) and vcd $\Gamma \leq 2$. To prove this we use the Lyndon exact sequence [23]

$$0 \to \mathbb{Z}[\Gamma/C] \to \mathbb{Z}[\Gamma]^n \to \mathbb{Z}[\Gamma] \to \mathbb{Z} \to 0,$$

where n is the number of generators in some 1-relator presentation of Γ, and C is a finite cyclic subgroup of Γ. It is known [19] that Γ is virtually torsion-free, and clearly the above exact sequence provides a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$ of length 2 for any torsion-free subgroup $\Gamma' \subseteq \Gamma$ of finite index, whence our assertion. We remark that it is easy to realize Lyndon's exact sequence topologically as the cellular chain complex of a 2-dimensional CW-complex on which Γ operates properly and with compact quotient.

4. If Γ is a finitely generated nilpotent group then Γ is of type (WFL) and vcd Γ is equal to the rank (or Hirsch number) of Γ.

5. $GL_n(\mathbb{Z})$ is of type (WFL) and has virtual cohomological dimension $n(n-1)/2$. This is, of course, a special case of the Borel-Serre results on arithmetic groups ([8], [32]), but we will indicate here a different proof due to Ash [1], based on the reduction theory of Voronoi [43]. Let X be the space of positive-definite real quadratic forms in n variables, modulo multiplication by positive scalars. The group $GL_n(\mathbb{Z})$ acts properly on X (but with non-compact quotient). We have $\dim X = \frac{n(n+1)}{2} - 1$. According to Voronoi [43] (see also [22] and the references cited there), X can be enlarged to a space X^* with the following properties:

(a) The action of $GL_n(\mathbb{Z})$ on X extends to X^*, and $X^*/GL_n(\mathbb{Z})$ is compact. (This extended action, however, is not proper.)

(b) X^* admits a cell-decomposition compatible with the action of $GL_n(\mathbb{Z})$.

Let $\partial X^* = X^* - X$. A glance at Voronoi's definition of the cells of X^* shows:

(c) ∂X^* is a subcomplex and contains the $(n-2)$-skeleton of X^*.

It is easy to see that X^* admits a barycentric subdivision compatible with the $GL_n(\mathbb{Z})$-action, and we denote by X' the 'simplicial complement' of ∂X^* in this subdivision, i. e. the union of all closed simplices which are disjoint from ∂X^*. Then X' inherits a simplicial action of $GL_n(\mathbb{Z})$, and this action is proper since $X' \subset X$. Moreover, $X = X^* - \partial X^*$ admits a canonical deformation retraction onto X' (cf. [34], ch. 3, sec. 3, proof of Cor. 11), so X' is contractible. From (a) we see that $X'/GL_n(\mathbb{Z})$ is compact (whence $GL_n(\mathbb{Z})$ is of type (WFL)), and from (c) we see that X' has codimension at least $n - 1$ in X^*, so that

$$\operatorname{vcd} GL_n(\mathbb{Z}) \leq \dim X' \leq \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2} .$$

Finally, to show that these inequalities are in fact equalities, we need only note that $GL_n(\mathbb{Z})$ contains the strict upper triangular group, which is a finitely generated nilpotent group of rank $n(n-1)/2$; thus $\operatorname{vcd} GL_n(\mathbb{Z}) \geq n(n-1)/2$.

Remark. The fact that X retracts onto a $GL_n(\mathbb{Z})$-invariant subspace X' of dimension $n(n-1)/2$ was first proved by Serre for $n = 2$ (cf. [32], or [31], Ch. I, 4. 2), by Soulé [33] for $n = 3$, and by Ash [1] for arbitrary n. More generally, Ash proves the analogous statement for a class of arithmetic groups including the groups $GL_n(\mathbb{Z})$, using a generalization of Voronoi's theory.

§3. Duality groups and virtual duality groups

References: [5], [6].

For any group Γ we may regard $\mathbb{Z}\Gamma$ as a left Γ-module and define $H^*(\Gamma, \mathbb{Z}\Gamma)$. (The group $H^1(\Gamma, \mathbb{Z}\Gamma)$, for example, arises in the theory of ends of groups.) Since $\mathbb{Z}\Gamma$ is also a right Γ-module and the left and right actions commute, the groups $H^i(\Gamma, \mathbb{Z}\Gamma)$ inherit a right Γ-module structure. These modules play a special role in the theory of groups of type (FP).

Definition. Γ is called a duality group if the following two conditions are satisfied:

(i) Γ is of type (FP).

(ii) There is an integer n such that $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$ and $H^n(\Gamma, \mathbb{Z}\Gamma)$ is $\mathbb{Z}$-torsion-free.

The integer n here is necessarily equal to $\operatorname{cd}\Gamma$; in fact, it is easy to see for any group Γ of type (FP) that $\operatorname{cd}\Gamma$ is equal to the largest integer i such that $H^i(\Gamma, \mathbb{Z}\Gamma) \neq 0$.

If Γ is a duality group then the Γ-module $D = H^n(\Gamma, \mathbb{Z}\Gamma)$ is called the dualizing module of Γ. The terminology 'duality group' and 'dualizing module' is justified by the existence of a duality isomorphism

$$(3.1)\quad H^i(\Gamma, M) \approx H_{n-i}(\Gamma, D \otimes M)$$

for any integer i and Γ-module M, where the tensor product is over $\mathbb{Z}$ and is given the diagonal Γ-action: $\gamma \cdot (d \otimes m) = d\gamma^{-1} \otimes \gamma m$ for $\gamma \in \Gamma$, $d \in D$, $m \in M$.

To prove 3.1, choose a finite projective resolution P of length n of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and let P' be the dual complex of projective right $\mathbb{Z}\Gamma$-modules, i.e. $P' = \operatorname{Hom}_{\mathbb{Z}\Gamma}(P, \mathbb{Z}\Gamma)$. Since $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$, P' provides a projective resolution of D over $\mathbb{Z}\Gamma$:

$$0 \to P'_0 \to \dots P'_n \to D \to 0.$$

Using the canonical isomorphism $\operatorname{Hom}_{\mathbb{Z}\Gamma}(P, M) \approx P' \otimes_{\mathbb{Z}\Gamma} M$, we deduce

$$H^i(\Gamma, M) \approx \operatorname{Tor}^{\mathbb{Z}\Gamma}_{n-i}(D, M).$$

Finally, since D is $\mathbb{Z}$-torsion-free we have

$$\operatorname{Tor}^{\mathbb{Z}\Gamma}_*(D, M) \approx H_*(\Gamma, D \otimes M),$$

whence 3.1.

Remarks. 1. Conversely, if Γ is a group such that there exist isomorphisms of the form 3.1 which are natural in M, (where D is a fixed Γ-module and n is a fixed integer), then Γ is a duality group. Indeed, Γ is then of type (FP) by [13] or [36], and condition (ii) above is easily derived from 3.1.

2. If the dualizing module D is $\mathbb{Z}$-free, which is the case in all known examples, then there are also isomorphisms

$$(3.2)\quad H_i(\Gamma, M) \approx H^{n-i}(\Gamma, \mathrm{Hom}(D, M)),$$

where $\mathrm{Hom}(\ ,\ ) = \mathrm{Hom}_{\mathbb{Z}}(\ ,\ )$, with the diagonal Γ-action.

3. If Γ is an arbitrary group of type (FP), then one can still derive isomorphisms of the form 3.1 and 3.2, but with the dualizing module D replaced by a 'dualizing chain complex'. (The groups on the right-hand side of 3.1 and 3.2 must then be interpreted as in the appendix at the end of this section.) Conversely, the existence of such generalized duality isomorphisms, natural in M, implies that Γ is of type (FP).

A duality group is said to be a Poincaré duality group if D, as $\mathbb{Z}$-module, is infinite cyclic. In this case the duality isomorphisms take a form more familiar to topologists:

$$H^i(\Gamma, M) \approx H_{n-i}(\Gamma, \tilde{M}),$$

where $\tilde{M}$ denotes M with the Γ-action 'twisted' by the character $\Gamma \to \{\pm 1\}$ by which Γ acts on D. (For example, if there exists a $K(\Gamma, 1)$ which is a closed manifold, then Γ is a Poincaré duality group.) From the point of view of group theory, however, Poincaré duality is rather rare. Torsion-free arithmetic groups, for example, are always duality groups, but they are Poincaré duality groups only in the rank 0 case ([8], 11.4).

A group Γ is said to be a virtual duality group if it contains a subgroup of finite index which is a duality group. This is equivalent to saying that Γ is of type (VFP) and that Γ satisfies condition (ii) of the definition of 'duality group'. Again we set $D = H^n(\Gamma, \mathbb{Z}\Gamma)$ and we note that every torsion-free subgroup $\Gamma' \subseteq \Gamma$ of finite index is a duality group whose dualizing module is D, regarded as Γ'-module. [More generally, if Γ is an arbitrary group of type (VFP) then one can find a chain complex of Γ-modules which serves as dualizing complex in the sense of Remark 3 above for every torsion-free subgroup of finite index.]

We mention one example, which is a special case of the Borel-Serre results on arithmetic groups [8]: The group $GL_r(\mathbb{Z})$ is a virtual duality

group of dimension $r(r-1)/2$, with

$$(3.3)\quad D = \begin{cases} St & \text{if } r \text{ is odd} \\ \tilde{St} & \text{if } r \text{ is even,} \end{cases}$$

where St is the 'Steinberg module' and $\tilde{St}$ denotes St with the $GL_r(\mathbb{Z})$-action twisted by $\det : GL_r(\mathbb{Z}) \to \{\pm 1\}$.

Appendix. Homology with coefficients in a chain complex

Let Γ be a group and $C = (C_i)_{i\geq 0}$ a chain complex of $\mathbb{Z}\Gamma$-modules. We then set

$$H_*(\Gamma, C) = H_*(P \otimes_{\mathbb{Z}\Gamma} C),$$

where P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and the tensor product is the total tensor product, i.e. the total complex associated to the double complex $P_\cdot \otimes_{\mathbb{Z}\Gamma} C_\cdot$. Note that if C consists of a single module M concentrated in dimension 0, then $H_*(\Gamma, C) = H_*(\Gamma, M)$.

The definition immediately gives us two spectral sequences converging to $H_*(\Gamma, C)$. The first has

$$E^2_{pq} = H_p(\Gamma, H_q C);$$

the second has

$$E^1_{pq} = H_q(\Gamma, C_p),$$

with the differential d^1 induced by the differential in $C_\cdot$. In particular, one obtains from these spectral sequences the following two properties of $H_*(\Gamma, C)$:

(3.4) If each C_p is projective over $\mathbb{Z}\Gamma$ then $H_*(\Gamma, C) \approx H_*(C_\Gamma)$. [Here $C_\Gamma = H_0(\Gamma, C) = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C$] More generally, the same conclusion holds if each C_p is H_*-acyclic, i.e. if $H_q(C_p) = 0$ for $q > 0$.

(3.5) If $f : C \to C'$ is a weak equivalence of chain complexes (i.e. $f_* : H_*C \to H_*C'$ is an isomorphism), then f induces an isomorphism

$H_*(\Gamma, C) \tilde{\to} H_*(\Gamma, C')$.

One can also define cohomology groups $H^*(\Gamma, C)$, where $C = (C^i)_{i \geq 0}$ is a cochain complex, as the cohomology of the total complex associated to $\mathrm{Hom}_{\mathbb{Z}\Gamma}(P_\cdot, C^\cdot)$. Again there are two spectral sequences and properties analogous to those above.

PART II. EULER CHARACTERISTICS

Main references: [11], [30]; see also [4].

We wish to define the Euler characteristic of a group of type (FP) as the alternating sum of the 'ranks' of the projective modules P_i which occur in a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. We begin, therefore, by defining a suitable notion of rank.

§4. Ranks of projective modules

If Γ is a group and P a Γ-module, we denote by P_Γ the abelian group $H_0(\Gamma, P) = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} P$. If P is finitely generated and projective over $\mathbb{Z}\Gamma$ then P_Γ is a finitely generated free $\mathbb{Z}$-module, and we set

$$\varepsilon(P) = \mathrm{rank}_{\mathbb{Z}}(P_\Gamma).$$

We will sometimes write $\varepsilon_\Gamma(P)$ instead of $\varepsilon(P)$ when this is necessary for clarity. The following proposition shows that ε has the multiplicative property which one expects of a reasonable 'rank':

(4.1) **Proposition.** *Let $\Gamma' \subset \Gamma$ be a subgroup of finite index. If P is a finitely generated projective $\mathbb{Z}\Gamma$-module, then P is also finitely generated and projective as $\mathbb{Z}\Gamma'$-module, and*

$$\varepsilon_{\Gamma'}(P) = (\Gamma : \Gamma') \cdot \varepsilon_\Gamma(P).$$

Proof. Let $\Gamma'' \subseteq \Gamma'$ be a subgroup of finite index which is normal in Γ. Then we may replace Γ, Γ', and P by Γ/Γ'', Γ'/Γ'', and $P_{\Gamma''}$ to reduce to the case where Γ is finite. But in this case one knows by a theorem of Swan that $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is free over $\mathbb{Q}\Gamma$. It is clear, then, that

$\varepsilon_\Gamma(P)$ is simply the rank of this free $\mathbb{Q}\Gamma$-module, and the proposition follows at once. (Proofs of Swan's theorem can be found in [38], [39], [2], and [3]; see also [4].)

Remark. There is another notion of rank, which we will denote $\rho(P)$, defined as the coefficient of the conjugacy class of 1 in the Hattori-Stallings rank of P. (Recall that the Hattori-Stallings rank of P, which we denote r(P), is a finite linear combination of Γ-conjugacy classes, cf. [3], [4].) The rank ρ, like ε, has the multiplicative property

$$(4.2)\quad \rho_{\Gamma'}(P) = (\Gamma : \Gamma') \cdot \rho_\Gamma(P).$$

Bass's 'weak conjecture' ([3], p. 156) says that one always has $\varepsilon = \rho$, and, as Bass observed ([3], 6.10), this is easily proved if Γ is residually finite. To see this, note first that one can express $\varepsilon(P)$ as the sum of the coefficients of r(P), hence $\varepsilon(P) = \rho(P)$ if r(P) is concentrated at the conjugacy class of 1. Now if Γ is residually finite, then we can find a subgroup Γ' of finite index which does not contain the finitely many non-trivial conjugacy classes where r(P) has a non-zero coefficient. We will then have $\varepsilon_{\Gamma'}(P) = \rho_{\Gamma'}(P)$, and hence $\varepsilon_\Gamma(P) = \rho_\Gamma(P)$ by 4.1 and 4.2.

§5. Euler characteristics for groups of type (FP)

One can use either of the ranks ε and ρ discussed in the previous section to define the Euler characteristic of a group of type (FP). For our purposes it will be more convenient to use ε. (Of course, the two definitions agree if Γ is residually finite by what we have just proved, and they agree for all Γ if Bass's weak conjecture is true. See [3], [4], and [17] for a discussion of the Euler characteristic based on ρ.)

Thus let Γ be of type (FP) and let $P = (P_i)$ be a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. We then set

$$\chi(\Gamma) = \sum(-1)^i \varepsilon(P_i) = \sum(-1)^i \operatorname{rank}_{\mathbb{Z}}(P_i)_\Gamma .$$

Note that the homology of the complex P_Γ is $H_*\Gamma$, so we can also write

$$\chi(\Gamma) = \sum(-1)^i \operatorname{rank}_{\mathbb{Z}}(H_i\Gamma) .$$

Thus $\chi(\Gamma)$ is simply the 'naive' Euler characteristic, which could have been defined a priori, without any discussion of ranks. The point of the definition in terms of ε, however, is that we immediately obtain from 4.1 the multiplicative property

(5.1) $\chi(\Gamma') = (\Gamma : \Gamma') \cdot \chi(\Gamma)$

if $\Gamma' \subset \Gamma$ is a subgroup of finite index. This property is by no means obvious from the naive definition, and some argument like that of §4 is needed in order to prove it. On the other hand, (5.1) is obvious if Γ is of type (FL). We will also need a multiplicative property of the Euler characteristic with respect to the coefficient module; again this is obvious if Γ is of type (FL) but requires some work in general.

(5.2) **Proposition.** *Suppose Γ is of type (FP), k is a field, and V is a $k\Gamma$-module of finite dimension over k. Then*

$$\sum(-1)^i \dim_k H_i(\Gamma, V) = \chi(\Gamma) \cdot \dim_k V = \sum(-1)^i \dim_k H^i(\Gamma, V).$$

A proof of the second equality can be found in [11], §4, and the first equality is proved similarly.

Before proceeding further, we mention a group theoretic application of the existence of an integer-valued Euler characteristic satisfying 5.1 for groups of type (FP):

(5.3) **Proposition.** *Let Γ be a group of type (FP). If Γ can be embedded as a subgroup of finite index in a torsion-free group $\overline{\Gamma}$, then $\chi(\Gamma)$ is divisible by $(\overline{\Gamma} : \Gamma)$.*

The proof is immediate, for $\overline{\Gamma}$ is of type (FP) by 1.3, hence

$$\frac{\chi(\Gamma)}{(\overline{\Gamma} : \Gamma)} = \chi(\overline{\Gamma}) \in \mathbb{Z} .$$

Thus $|\chi(\Gamma)|$, if non-zero, provides an obstruction to the existence of torsion-free enlargements of Γ.

(5.4) **Remark.** Even if one is only interested in the case where Γ is of type (FL), the proof requires a theory of Euler characteristics for

groups of type (FP), since one does not know that $\overline{\Gamma}$ will be of type (FL).

(5.5) Corollary. *Let $1 \to \Gamma \to E \to P \to 1$ be a group extension, where Γ is of type (FP) and P has prime order p. If $p \nmid \chi(\Gamma)$ then the extension splits.*

In fact, E necessarily has torsion by 5.3; Γ being torsion-free, it follows that any non-trivial finite subgroup of E must map isomorphically to P, thus providing a splitting.

We will see later (Cor. 7.3) that 5.5 can be substantially improved.

§6. Extension to groups of type (VFP)

Let Γ be a group of type (VFP). Following the method of Wall [44], we then define $\chi(\Gamma)$ by choosing a subgroup Γ' of finite index which is of type (FP) and setting

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{(\Gamma : \Gamma')} ,$$

the right-hand side being independent of the choice of Γ' by 5.1. Note that $\chi(\Gamma)$ is a rational number and is not, in general, an integer. For example, if Γ is finite then $\chi(\Gamma) = 1/|\Gamma|$. If Γ is torsion-free, on the other hand, then Γ is of type (FP) and hence $\chi(\Gamma) \in \mathbb{Z}$.

We list some useful properties of the Euler characteristic:

(6.1) *If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then*

$$\chi(\Gamma') = (\Gamma : \Gamma') \cdot \chi(\Gamma).$$

This is immediate from the definition.

(6.2) *Let $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ be a group extension, where Γ' and Γ'' are of type (VFP). If Γ is virtually torsion-free then Γ is of type (VFP) and*

$$\chi(\Gamma) = \chi(\Gamma') \cdot \chi(\Gamma'').$$

The proof that Γ is of type (VFP) is straightforward. To prove the Euler characteristic formula, one reduces to the case where all

groups are of type (FP), in which case the result follows from a spectral sequence argument together with 5.2.

(6.3) Let Γ be an amalgamation $\Gamma_1 *_A \Gamma_2$ where $A \hookrightarrow \Gamma_i$, and suppose Γ_1, Γ_2, and A are of type (VFP). If Γ is virtually torsion-free then Γ is of type (VFP) and

$$\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2) - \chi(A).$$

This can be proved exactly as in [30], where the (WFL) case is treated.

As an example of 6.3 we may take $\Gamma = SL_2(\mathbb{Z}) \approx \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$). We obtain

$$\chi(SL_2(\mathbb{Z})) = \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{2} = -\tfrac{1}{12}.$$

(Alternatively, one can derive this formula from the fact that $SL_2(\mathbb{Z})$ contains a subgroup of index 12 which is free on two generators, cf. [30], 1.8, Ex. 2).

The theory of Euler characteristics becomes especially interesting when applied to Chevalley groups over a ring of algebraic integers. In this case one has Harder's formula expressing $\chi(\Gamma)$ in terms of values of ζ-functions (see [20], [30], [32]). For future reference we record two special cases of this formula:

$$(6.4)\quad \chi(Sp_{2n}(\mathbb{Z})) = \prod_{i=1}^{n} \zeta(1 - 2i) = \prod_{i=1}^{n} -B_{2i}/2i,$$

where B_{2i} is the $2i^{\text{th}}$ Bernoulli number ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ...).

$$(6.5)\quad \chi(E_7(\mathbb{Z})) = -\frac{691 \cdot 43867}{2^{21} \cdot 3^9 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 19}.$$

Note that $Sp_2 = SL_2$, so we recover from 6.4 (with $n = 1$) the formula $\chi(SL_2(\mathbb{Z})) = -1/12$.

§7. Integrality properties of $\chi(\Gamma)$

Throughout this section Γ will denote an arbitrary group of type (VFP). We have seen that $\chi(\Gamma)$ need not be an integer if Γ has torsion.

The results of this section and the next resulted from an attempt to explain more precisely the relation between the torsion in Γ and the non-integrality of $\chi(\Gamma)$. The first result along these lines is the following observation due to Serre ([30], 1. 8, Prop. 13):

(7. 1) **Proposition.** *If p is a prime such that Γ has no p-torsion, then $\chi(\Gamma)$ is p-integral, i. e. p does not occur in the denominator of $\chi(\Gamma)$.*

[Taking $\Gamma = E_7(\mathbb{Z})$, for example, it follows from this proposition and 6. 5 that $E_7(\mathbb{Z})$ must have p-torsion for $p = 2, 3, 5, 7, 11, 13, 19.$]

To prove the proposition choose a torsion-free normal subgroup $\Gamma' \subseteq \Gamma$ of finite index, and choose $\Gamma_p (\Gamma' \subseteq \Gamma_p \subseteq \Gamma)$ so that Γ_p/Γ' is a p-Sylow subgroup of Γ/Γ'. Then $(\Gamma : \Gamma_p)$ is relatively prime to p, and Γ_p is torsion-free (since any torsion would be p-torsion). Thus $\chi(\Gamma_p) \in \mathbb{Z}$ and $\chi(\Gamma) = \chi(\Gamma_p)/(\Gamma : \Gamma_p)$ is indeed p-integral.

Serre went on to conjecture the following more precise result, which was proved in [11]:

(7. 2) **Theorem.** *Let m be the least common multiple of the orders of the finite subgroups of Γ. Then $m \cdot \chi(\Gamma) \in \mathbb{Z}$.*

Note that the p-part of m for a given prime p is simply the maximal order of a p-subgroup of Γ, so the theorem can be restated as follows: If a prime power p^k occurs in the denominator of $\chi(\Gamma)$, then Γ has a subgroup of order p^k.

For example, taking $\Gamma = E_7(\mathbb{Z})$ again, we see that not only must $E_7(\mathbb{Z})$ contain elements of order 2, 3, ..., but it must contain subgroups of order 2^{21}, 3^9, ... This application of Theorem 7. 2 to the study of torsion in the exceptional Chevalley groups is due to Serre. See [32] for a more detailed discussion.

Another application is the promised improvement of 5. 5.

(7. 3) **Corollary.** *Let $1 \to \Gamma \to E \to P \to 1$ be a group extension such that Γ is of type (FP) and P is a p-group for some prime p. If $p \nmid \chi(\Gamma)$ then the extension splits.*

Proof. One has $\chi(E) = \chi(\Gamma)/|P|$, and this fraction is in lowest terms. By the theorem, E must contain a subgroup of order equal to $|P|$, and any such subgroup provides a splitting of the extension.

As an example of the corollary, take $\Gamma = F_n$, the free group on n generators. Then $\chi(F_n) = 1 - n$, so an extension as above must split if $p \nmid n - 1$. This result is vacuous if $n = 1$ and easy to prove directly if $n = 2$, using the known structure of the group of outer automorphisms of F_2 ([24], §3.5, Cor. N4). If $n \geq 3$, however, I know of no proof other than that given here, based on the theory of Euler characteristics.

We now prove Theorem 7.2. Let X be a finite-dimensional contractible simplicial complex on which Γ acts properly (2.1). Let $\Gamma' \subseteq \Gamma$ be a torsion-free normal subgroup of finite index and let $Y = X/\Gamma'$. [Note: Replacing X by its barycentric subdivision, if necessary, we can assume that Y inherits a cell-decomposition from that of X. Taking another barycentric subdivision, we can even make Y simplicial, cf. [10].] Since Γ' is of type (FP) and acts freely on X, we have $\chi(\Gamma') = \chi(Y)$, the latter being, by definition, $\sum(-1)^i \mathrm{rk}(H_i Y)$. Hence $\chi(\Gamma) = \chi(Y)/(\Gamma : \Gamma')$ and what we are trying to prove, then, is that

$$\frac{m}{(\Gamma : \Gamma')} \cdot \chi(Y) \in \mathbb{Z},$$

or, in other words, that $\chi(Y)$ is divisible by the integer $d = (\Gamma : \Gamma')/m$.

To this end we note that the action of Γ on X induces a (simplicial) action of $G = \Gamma/\Gamma'$ on $Y = X/\Gamma'$. Moreover, the isotropy groups G_y $(y \in Y)$ are simply the images in G of the isotropy groups Γ_x $(x \in X)$, hence they all have order dividing m. Thus every orbit Gy has cardinality divisible by d, and one would like to conclude that $\chi(Y)$ is divisible by d. This is trivially true if Y is compact, since $\chi(Y)$ can then be computed by counting simplices, and the number of these in each dimension is divisible by d. If Y is not compact, one still knows that Y is finite dimensional and that H_*Y is finitely generated, and it turns out that these finiteness conditions on Y are enough to yield the result that $d \mid \chi(Y)$. In fact, one can prove:

(7.4) **Theorem.** Let Y be a paracompact space of finite cohomological dimension in the sense of sheaf theory, and assume that

$H^*(Y, \mathbb{Z})$ is finitely generated. If a finite group G acts on Y and the cardinality of each orbit Gy is divisible by some integer d, then $d \mid \chi(Y)$.

[Here $H^*(Y, \mathbb{Z})$ denotes the sheaf-theoretic (or Čech) cohomology of Y, and $\chi(Y)$ is defined to be $\sum(-1)^i \mathrm{rk}(H^i(\Gamma, \mathbb{Z}))$.]

We will sketch the proof of this theorem; for further details see [11], §2. Note first that, by a Sylow argument, we may reduce to the case where G is a p-group for some prime p. Moreover, since $H^*(Y, \mathbb{Z})$ is finitely generated, $\chi(Y)$ is equal to the mod p Euler characteristic $\sum(-1)^i \dim_{\mathbb{Z}_p} H^i(Y, \mathbb{Z}_p)$. Throughout the remainder of this proof, then, $H^*(\ )$ will denote $H^*(\ , \mathbb{Z}_p)$ and χ will denote the mod p Euler characteristic.

(a) If G acts freely on Y, then the desired result that $|G| \mid \chi(Y)$ is a well-known consequence of Smith theory, cf. [7], ch. III. More generally, one has the following relative version of this result: If $Y' \subset Y$ is a G-invariant closed subspace such that $H^*(Y, Y')$ is finitely generated and G acts freely in $Y - Y'$, then $|G| \mid \chi(Y, Y')$.

(b) In the general case we use the technique of 'stratification by orbit type'. For any subgroup $H \subseteq G$ let $Y_H = \{y \in Y : G_y = H\}$ and let $Y_{\{H\}} = G \cdot Y_H$. (Thus $Y_{\{H\}}$ is the union of all orbits of type G/H.) Let $\mathcal{C}$ be a set of representatives for the conjugacy classes of subgroups of G which occur as isotropy groups in Y. It is easy to see that there is a filtration of Y by closed subspaces $\emptyset = Y_0 \subset \ldots \subset Y_n = Y$, such that the successive differences $Y_i - Y_{i-1}$ are the subspaces $Y_{\{H\}}$ $(H \in \mathcal{C})$. It follows that

$$\chi(Y) = \sum_{H \in \mathcal{C}} \chi'(Y_{\{H\}}),$$

where χ' is defined as follows: If A is a locally closed subspace of Y then we write $A = B - B'$, where B and B' are closed and $B' \subseteq B$; if $H^*(B, B')$ is finitely generated then we set $\chi'(A) = \chi(B, B')$, this being independent of the choice of (B, B'). (Alternatively, $\chi'(A)$ can be defined in terms of the cohomology of A with supports in the family of subsets of A which are closed in Y.) One must verify, of course, that $\chi'(Y_{\{H\}})$ is defined, but this follows easily from the fact (known

from Smith theory) that each fixed point set X^H has finitely generated mod p cohomology.

It suffices, therefore, to prove that $\chi'(Y_{\{H\}})$ is divisible by $(G : H)$. Now clearly

$$Y_{\{H\}} = \coprod_{g \in G/N(H)} g \cdot Y_H,$$

where $N(H)$ is the normalizer of H in G, so

$$\chi'(Y_{\{H\}}) = (G : N(H)) \cdot \chi'(Y_H).$$

On the other hand, the group $N(H)/H$ acts freely in Y_H, so the relative version of (a) implies that $\chi'(Y_H)$ is divisible by $(N(H) : H)$. Thus $\chi'(Y_{\{H\}})$ is indeed divisible by $(G : N(H)) \cdot (N(H) : H) = (G : H)$.

§8. Formulas for $\chi(\Gamma)$

A careful examination of the proof of Theorem 7.2 yields more precise information than what was stated. For example, suppose Γ satisfies the following condition:

(8.1) Γ has only finitely many conjugacy classes of finite subgroups, and for each finite subgroup H the normalizer $N(H)$ is of type (VFP).

One can then derive ([11], §6) a formula of the form

$$(8.2)\quad \chi(\Gamma) = \tilde{\chi}(\Gamma) + \sum_{H \in \mathcal{C}} c_H / |H| ,$$

where $\tilde{\chi}(\Gamma)$ is the 'naive' Euler characteristic $\sum (-1)^i \mathrm{rk}_{\mathbf{Z}}(H_i \Gamma)$, $\mathcal{C}$ is a set of representatives for the conjugacy classes of non-trivial finite subgroups of Γ, and c_H is an integer which is defined in terms of the conjugation action of $N(H)$ on the ordered set of finite subgroups of Γ containing H. This formula then 'explains', in terms of the torsion in Γ, the failure of $\chi(\Gamma)$ to equal the integer $\tilde{\chi}(\Gamma)$. We will not prove 8.2 here, but we will instead give some results which are less precise but easier to use in practice.

We will need the notion of 'equivariant Euler characteristic' for a

pair (Γ, K), where Γ is a group and K a CW-complex on which Γ acts. For simplicity we will assume that the following two conditions are satisfied:

(i) The Γ-action permutes the cells of K.

(ii) For each cell σ of K, the isotropy group Γ_σ fixes σ pointwise.

We then say that K is an admissible Γ-complex. [Note: Condition (ii) is harmless in practice; in the case of a simplicial action, for example, it can always be achieved by passing to the barycentric subdivision.] If, in addition, K/Γ is compact and each isotropy group Γ_σ is of type (VFP), then we define the equivariant Euler characteristic $\chi_\Gamma(K)$ by

$$\chi_\Gamma(K) = \sum(-1)^{\dim \sigma}\chi(\Gamma_\sigma),$$

where σ ranges over a set of representatives for the cells of K mod Γ. It is easy to verify (cf. [30], 1.8, proof of Prop. 14(b)), that

$$\chi_{\Gamma'}(K) = (\Gamma : \Gamma') \cdot \chi_\Gamma(K)$$

if $\Gamma' \subset \Gamma$ is a subgroup of finite index.

We will be particularly interested in the case where K arises from a partially ordered set S on which Γ operates, i.e. K is the simplicial complex $K(S)$ (sometimes called the nerve of S) whose vertices are the elements of S and whose n-simplices correspond to the chains $s_0 < s_1 < \ldots < s_n$ in S. In this case we set

$$\chi_\Gamma(S) = \chi_\Gamma(K(S)),$$

if the right-hand side is defined.

We can now state (cf. [11], §6):

(8.3) **Theorem.** *Let Γ be a group which satisfies condition 8.1 and let $\mathfrak{F}$ be the set of non-trivial finite subgroups of Γ. Regard $\mathfrak{F}$ as an ordered set under inclusion, with Γ-action by conjugation. Then $\chi_\Gamma(\mathfrak{F})$ is defined and*

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathfrak{F}) \pmod{\mathbb{Z}}.$$

This theorem can be regarded as a formula for the 'fractional part' of $\chi(\Gamma)$ in terms of the Euler characteristics of groups of the form $N(H_0) \cap \ldots \cap N(H_n)$, where $H_0 \subset \ldots \subset H_n$ is a chain of non-trivial finite subgroups of Γ. There is also a 'local' version of the theorem, proved in [12], which says that if we just want the 'p-fractional part' of $\chi(\Gamma)$ for a fixed prime p, then it suffices to consider those finite subgroups H which are p-groups, i.e.

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathcal{F}_p) \pmod{\mathbb{Z}_{(p)}},$$

where $\mathcal{F}_p$ is the set of non-trivial finite p-subgroups of Γ and $\mathbb{Z}_{(p)}$ is $\mathbb{Z}$ localized at p. Quillen [26] improved this result by showing that $\mathcal{F}_p$ can be replaced by the smaller set $\mathcal{A}_p$ consisting of the non-trivial elementary abelian p-subgroups of Γ. (Recall that an elementary abelian p-group is a group isomorphic to $(\mathbb{Z}/p)^r$ for some integer r, called the rank of the group.) The precise statement of this improved result is:

(8.4) **Theorem.** *Let Γ be a group and p a prime such that $N(H)$ is of type (VFP) for every elementary abelian p-subgroup $H \subseteq \Gamma$. Then $\chi_\Gamma(\mathcal{A}_p)$ is defined and*

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathcal{A}_p) \pmod{\mathbb{Z}_{(p)}}.$$

We will give the proofs of Theorems 8.3 and 8.4 in the next section.

Remark. Theorem 8.4 (unlike Theorem 8.3) is non-vacuous even if Γ is finite. In this case the congruence above can be unscrambled to yield

$$\chi(\mathcal{A}_p) \equiv 1 \pmod{p^k},$$

where p^k is the highest power of p dividing $|\Gamma|$ and $\chi(\mathcal{A}_p)$ is the Euler characteristic of the finite complex $K(\mathcal{A}_p)$. See Quillen [26] for further results about the homotopy type of $K(\mathcal{A}_p)$.

The simplest case of Theorem 8.4 is that where every elementary abelian p-subgroup of Γ has rank ≤ 1, i.e. Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. In this case $K(\mathcal{A}_p)$ is discrete and one has

$$\chi_\Gamma(\mathcal{C}_p) = \sum \chi(N(P)) ,$$

where P ranges over the subgroups of Γ of order p, up to conjugacy. Using the fact that each P contains exactly $p - 1$ elements of order p, one can easily rewrite the right-hand side of this equation in terms of the elements of Γ of order p and their centralizers, and one obtains:

(8.5) Corollary. *Let Γ be a group of type (VFP) and p a prime such that Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. For each element α of Γ of order p, assume that the centralizer $Z(\alpha)$ is of type (VFP). Then Γ has only finitely many conjugacy classes of elements of order p, and*

$$\chi(\Gamma) \equiv \frac{1}{p-1} \sum \chi(Z(\alpha)) \qquad (\mathrm{mod}\ \mathbb{Z}_{(p)}) ,$$

where α ranges over the elements of order p, up to conjugacy.

As an application of this corollary, due to Serre, one can recover Kummer's criterion in terms of Bernoulli numbers for the irregularity of a prime p. This is done by taking $\Gamma = Sp_{p-1}(\mathbb{Z})$ and combining the above congruence with Harder's formula 6.4. See [11], §9.4, and [12], §4, for details and a generalization.

§9. Proofs of Theorems 8.3 and 8.4

The proofs will require the rudiments of equivariant homology theory. Specifically, we will need to know that there are groups $H_*^\Gamma(K)$, defined, say, if K is an admissible Γ-complex, and having the following three properties:

(9.1) If Γ acts freely on K then $H_*^\Gamma(K) \approx H_*(K/\Gamma)$.

(9.2) If $f : K \to K'$ is a Γ-equivariant cellular map which induces an isomorphism $H_*K \to H_*K'$, then f induces an isomorphism

$$H_*^\Gamma(K) \xrightarrow{\approx} H_*^\Gamma(K') .$$

(9.3) There is a spectral sequence converging to $H_*^\Gamma(K)$, with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma),$$

where Σ_p is a set of representatives for the p-cells of K mod Γ. Consequently, if K/Γ is compact and each Γ_σ is of type (FP), then

$$\sum (-1)^i \operatorname{rk}_{\mathbb{Z}} H_i^\Gamma(K) = \chi_\Gamma(K) .$$

There are various ways to define the equivariant homology groups and prove the above properties. For example, one can set

$$H_*^\Gamma(K) = H_*(\Gamma, C(K)) ,$$

where $C(K)$ is the cellular chain complex of K and the right-hand side is to be interpreted in the sense of the appendix to §3. The properties 9.1-9.3 then follow from results stated in that appendix.

We can now prove Theorem 8.3. First, the fact that $\chi_\Gamma(\mathcal{F})$ is defined is an easy consequence of 8.1, cf. [11], §5, Lemma. Now let X, as in the proof of Theorem 7.2, be a finite-dimensional contractible simplicial complex on which Γ acts properly, let $\Gamma' \subseteq \Gamma$ be a torsion-free normal subgroup of finite index, and let Y be the Γ/Γ'-complex X/Γ'. Assume further that X has been chosen so that X^H is contractible for $H \in \mathcal{F}$, cf. 2.2. Let X_0 be the set of points of X with non-trivial isotropy group and let $Y_0 = X_0/\Gamma'$. I claim that Y_0 has finitely generated homology. Accepting this for the moment, and noting that Γ/Γ' acts freely in $Y - Y_0$, we obtain (cf. proof of Theorem 7.4)

$$\chi(Y) \equiv \chi(Y_0) \pmod{(\Gamma : \Gamma')}.$$

Thus

$$(9.4)\quad \chi(\Gamma) = \frac{\chi(Y)}{(\Gamma : \Gamma')} \equiv \frac{\chi(Y_0)}{(\Gamma : \Gamma')} \pmod{\mathbb{Z}} .$$

Observe now that $X_0 = \cup_{H \in \mathcal{F}} X^H$. Since each X^H is contractible, one deduces that X_0 is homotopy equivalent to the 'nerve' of the covering $\{X^H\}$, and in the present context 'nerve' can be taken to mean the complex $K(\mathcal{F})$:

(9.5) $X_0 \simeq K(\mathcal{F})$.

(Cf. [11], Appendix B, and [26], proof of 4.1.) Moreover, this homotopy equivalence can be taken to be compatible with the Γ-action, in the sense that there is a third Γ-complex which maps to both X_0 and $K(\mathcal{F})$ by Γ-equivariant maps which are homotopy equivalences. Using 9.1 and 9.2, we conclude that

$$H_*(Y_0) \approx H_*^{\Gamma'}(X_0) \approx H_*^{\Gamma'}(K(\mathcal{F})) .$$

Thus $H_*(Y_0)$ is indeed finitely generated and, by 9.3, $\chi(Y_0) = \chi_{\Gamma'}(\mathcal{F})$; the right-hand side of 9.4 is therefore equal to $\chi_\Gamma(\mathcal{F})$, and the proof is complete.

Theorem 8.4 will be deduced from:

(9.6) Proposition. Let Γ be a group of type (VFP) and let K be an admissible Γ-complex such that $\chi_\Gamma(K)$ is defined. If p is a prime such that K^H is contractible for every non-trivial finite p-subgroup $H \subseteq \Gamma$, then

$$\chi_\Gamma(K) \equiv \chi(\Gamma) \qquad (\mathrm{mod}\ \mathbb{Z}_{(p)}).$$

Proof. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index. Replacing Γ by a subgroup Γ_p such that Γ/Γ_p is a p-Sylow subgroup of Γ/Γ', we may reduce to the case where Γ/Γ' is a p-group, in which case we will prove

$$\chi_\Gamma(K) \equiv \chi(\Gamma) \qquad (\mathrm{mod}\ \mathbb{Z}).$$

Note that every finite subgroup of Γ is now a p-group, so our hypothesis says that K^H is contractible for every $H \in \mathcal{F}$. We may therefore argue as in the proof above to deduce

$$\chi(\Gamma) = \frac{\chi(Y)}{(\Gamma : \Gamma')} \equiv \frac{\tilde{\chi}_{\Gamma'}(\mathcal{F})}{(\Gamma : \Gamma')} \qquad (\mathrm{mod}\ \mathbb{Z}) ,$$

where $\tilde{\chi}_{\Gamma'}(\mathcal{F}) = \sum(-1)^i \mathrm{rk}_{\mathbb{Z}_p} H_i^{\Gamma'}(K(\mathcal{F}), \mathbb{Z}_p)$. (One needs to use here the fact that $H_*(Y_0, \mathbb{Z}_p)$ is finitely generated by Smith theory, cf. proof of Theorem 7.4.) On the other hand, we may apply the same argument

with X replaced by $\overline{X} = X \times K$, since $\overline{X}^H = X^H \times K^H$ is still contractible for $H \in \mathcal{F}$. Writing $\overline{Y} = \overline{X}/\Gamma'$, we find

$$\chi_\Gamma(K) = \frac{\chi_{\Gamma'}(K)}{(\Gamma : \Gamma')} = \frac{\chi(\overline{Y})}{(\Gamma : \Gamma')} \equiv \frac{\tilde{\chi}_{\Gamma'}(\mathcal{F})}{(\Gamma : \Gamma')} \qquad (\mathrm{mod}\ \mathbf{Z}),$$

whence the proposition.

Proof of Theorem 8.4. We remark first that Γ has only finitely many conjugacy classes of p-subgroups. This follows from the fact that, with the notation we have been using, $H_*(Y^P, \mathbf{Z}_p)$ is finitely generated (and hence Y^P has only finitely many connected components) for every p-subgroup $P \subseteq \Gamma/\Gamma'$; see [15], proof of Lemma 4.11(a), or [25], proof of Prop. 14.5, for more details. In particular, Γ has only finitely many conjugacy classes of elementary abelian p-subgroups, and it follows easily that $\chi_\Gamma(\mathcal{A}_p)$ is defined. The theorem will now follow from Proposition 9.6 applied with $K = K(\mathcal{A}_p)$, if we verify that $K(\mathcal{A}_p)^H$ is contractible for each non-trivial p-subgroup $H \subseteq \Gamma$. Fix a central subgroup C of H of order p. If $A \in \mathcal{A}_p^H$, i.e. A is a non-trivial elementary abelian p-subgroup of Γ normalized by H, then A^H is non-trivial; hence we have a sequence of inclusions $A \supseteq A^H \subseteq C \cdot A^H \supseteq C$ in $\mathcal{A}_p^H$ and this yields the required contracting homotopy of $K(\mathcal{A}_p)^H = K(\mathcal{A}_p^H)$, cf. [26], 4.4.

PART III. FARRELL COHOMOLOGY THEORY

References: [18], [14].

Let Γ be an arbitrary group of finite virtual cohomological dimension. If Γ is torsion-free then $\mathrm{cd}\,\Gamma < \infty$ and therefore $H^*(\Gamma) = 0$ in high dimensions. This suggests (by analogy with the results of §§7 and 8) that, in general, one might try to 'explain' the high-dimensional cohomology of Γ in terms of the torsion in Γ. For this purpose it is convenient to use a modified cohomology theory $\hat{H}$ introduced by Farrell [18]. There is a map $H^i(\Gamma) \to \hat{H}^i(\Gamma)$ which is an isomorphism for $i > \mathrm{vcd}\,\Gamma$, and one has $\hat{H}^*(\Gamma) = 0$ if Γ is torsion-free. Thus it is reasonable to expect that, in some sense, $\hat{H}^*(\Gamma)$ isolates the cohomological contribution of the finite subgroups of Γ. It is not yet clear to what extent the Farrell theory will be useful in the study of the low-dimensional cohomology of Γ (which is

often more interesting than the high-dimensional cohomology, e.g. for applications to algebraic K-theory), but at the very least it allows one to break the study of $H^*(\Gamma)$ into two steps: (a) understand $\hat{H}^*(\Gamma)$; (b) understand the map $H^*(\Gamma) \to \hat{H}^*(\Gamma)$.

Farrell's cohomology theory is a generalization of the Tate cohomology theory for finite groups. We will therefore begin by reviewing the latter (§10); then in §§11 and 12 we discuss the foundations of Farrell's theory. Two of the well-known applications of Tate cohomology theory are the Nakayama-Rim theory of cohomologically trivial modules (cf. [29]) and the theory of groups with periodic cohomology (cf. [16]); in §§13 and 14 we give the generalizations of these theories to infinite groups, using Farrell cohomology. Finally, §15 contains the results alluded to above, relating $\hat{H}^*(\Gamma)$ to the finite subgroups of Γ; as an application, we obtain some results on $H^*(SL_3(\mathbb{Z}[\frac{1}{2}]))$.

§10. Review of Tate cohomology theory

Let G be a finite group. To define the Tate groups $\hat{H}^*(G)$, one begins with a projective resolution $P = (P_i)_{i \geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ and 'completes' it to a complex of projectives $\hat{P}$ which is acyclic in all dimensions:

$$\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots .$$

The existence of such a completion is easily proved as follows. To begin, one chooses an injection $i : \mathbb{Z} \hookrightarrow P_{-1}$ of $\mathbb{Z}G$-modules, such that P_{-1} is projective and i is $\mathbb{Z}$-split (e.g. take $P_{-1} = \mathbb{Z}G$ and $i(1) = N = \sum_{g \in G} g$). Let $C = \operatorname{coker} i$. Since C is $\mathbb{Z}$-free, one can find a $\mathbb{Z}$-split injection $j : C \hookrightarrow P_{-2}$, where P_{-2} is projective (e.g. take $P_{-2} = \mathbb{Z}G \otimes C$, with G acting on the first factor, and let $j(c) = \sum_{g \in G} g \otimes g^{-1}c$). Continuing in this way we obtain $\hat{P}$:

$$\ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \ldots .$$

$$\mathbb{Z} \qquad C$$

It is easy to see that any two such completions are canonically homotopy equivalent, and we can therefore define

$$\hat{H}^*(G, M) = H^*(\mathrm{Hom}_{\mathbb{Z}G}(\hat{P}, M))$$

for any G-module M. The Tate theory has the following properties:

(10.1) $\hat{H}^*(G, M) = 0$ if G is the trivial group.

(10.2) As in ordinary cohomology theory one has long exact cohomology sequences associated to short exact sequences of modules, Shapiro's lemma, restriction and transfer maps, and cup products.

(10.3) $\hat{H}^*(G, M) = 0$ if M is an induced module $\mathbb{Z}G \otimes A$ for some abelian group A; hence the functors $\hat{H}^i(G, -)$ are effaceable and coeffaceable.

(10.4) $\hat{H}^i = H^i$ for $i > 0$.

(10.5) $\hat{H}^0$ is a quotient of H^0, namely, the cokernel of the norm map $N : H_0 \to H^0$.

(10.6) $\hat{H}^{-1}$ is a subgroup of H_0, namely, the kernel of the norm map $N : H_0 \to H^0$.

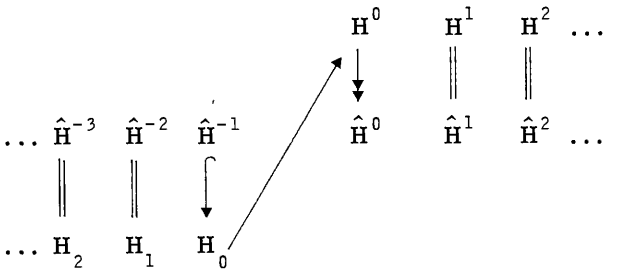
(10.7) $\hat{H}^i = H_{-i-1}$ if $i < -1$.

Properties 10.1-10.5 are easy to verify directly from the definition, while 10.6 and 10.7 follow from the fact that a complete resolution can be constructed by splicing together a finite type resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ with its dual $P' = \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G) \approx \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$:

(10.8) $$\cdots \to P_1 \to P_0 \to P'_0 \to P'_1 \to \cdots .$$

$$\mathbb{Z}$$

Thus we have a cohomology theory $\{\hat{H}^i\}$ consisting of the functors H^i and H_i for $i > 0$, together with modified H^0 and H_0 functors:



§11. Definition of $\hat{H}^*(\Gamma)$

Let Γ be a group such that $\operatorname{vcd}\Gamma = n < \infty$. (The previous section treated the case $n = 0$.) Let P be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. By a completion of P we will mean an acyclic complex $\hat{P}$ of projectives which agrees with P in sufficiently high dimensions. A completion of P can be constructed as follows: Let $K = \operatorname{Im}\{P_n \to P_{n-1}\}$. If $\Gamma' \subseteq \Gamma$ is a torsion-free subgroup of finite index, then K is $\mathbb{Z}\Gamma'$-projective, hence we can find an embedding $i : K \hookrightarrow \hat{P}_{n-1}$ where $\hat{P}_{n-1}$ is $\mathbb{Z}\Gamma$-projective and i is $\mathbb{Z}\Gamma'$-split (e.g. take $\hat{P}_{n-1} = \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} K$ and $i(x) = \sum \gamma \otimes \gamma^{-1}x$, where γ ranges over a set of representatives for the cosets Γ/Γ'). Applying the same process to coker i and continuing as in the previous section, we obtain a completion of P:

$$\cdots \to P_{n+1} \to P_n \to \hat{P}_{n-1} \to \cdots \quad .$$

$$\searrow \quad \nearrow$$

$$K$$

In case Γ is of type (VFP), we can also use the following method for constructing complete resolutions, which generalizes the splicing construction (10.8) available if Γ is finite: Take the original resolution P to be of finite type and let P' be the dual complex $\operatorname{Hom}_{\mathbb{Z}\Gamma}(P, \mathbb{Z}\Gamma)$. One can show that there exists a chain complex $Q = (Q_i)_{i \geq 0}$ of finitely generated projectives which maps to P' by a weak equivalence of the form

$$\begin{array}{ccccccccccc} \cdots & \to & Q_{n+1} & \to & Q_n & \to \cdots \to & Q_0 & \to & 0 & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \to & 0 & \to & P'_0 & \to \cdots \to & P'_n & \to & P'_{n+1} & \to \cdots \end{array}.$$

(If Γ is a virtual duality group, for example, then Q is simply a finite type projective resolution of the module $D = H^n(\Gamma, \mathbb{Z}\Gamma)$.) The mapping cone of this weak equivalence is then an acyclic complex of projectives, whose dual is the desired completion $\hat{P}$.

Returning to the general case, now, one shows that any two completions are canonically homotopy equivalent, hence we can define the Farrell cohomology groups by

$$\hat{H}^*(\Gamma, M) = H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(\hat{P}, M)).$$

One shows also that there is a chain map $\hat{P} \to P$, well-defined up to homotopy, whence a map

$$H^*(\Gamma, M) \to \hat{H}^*(\Gamma, M).$$

We will often suppress the coefficient module M from the notation and simply write $\hat{H}^*(\Gamma)$.

The Farrell theory has properties analogous to the properties of the Tate theory listed in §10.

(11.1) $\hat{H}^*(\Gamma) = 0$ if Γ is torsion-free.

(11.2) One has long exact cohomology sequences, Shapiro's lemma, restriction and transfer maps, and cup products. Moreover, there is a 'Hochschild-Serre' spectral sequence associated to a short exact sequence $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ of groups of finite virtual cohomological dimension, provided either Γ' or Γ'' is torsion-free. If Γ'' is torsion-free this takes the form

$$E_2^{pq} = H^p(\Gamma'', \hat{H}^q(\Gamma')) \Rightarrow \hat{H}^{p+q}(\Gamma),$$

and if Γ' is torsion-free then it takes the form

$$E_2^{pq} = \hat{H}^p(\Gamma'', H^q(\Gamma')) \Rightarrow \hat{H}^{p+q}(\Gamma).$$

(Note, in particular, that the edge homomorphism of the latter spectral sequence yields an inflation map $\hat{H}^*(\Gamma'') \to \hat{H}^*(\Gamma)$ for any Γ''-module of coefficients in the case where Γ' is torsion-free.)

(11.3) $\hat{H}(\Gamma, M) = 0$ if M is an induced module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} M'$, where Γ' is a torsion-free subgroup of finite index and M' is a Γ'-module; hence the functors $\hat{H}^i(\Gamma, -)$ are effaceable and co-effaceable.

(11.4) $\hat{H}^i = H^i$ for $i > n = \operatorname{vcd} \Gamma$.

(11.5) $\hat{H}^n(\Gamma, M)$ is isomorphic to the cokernel of the transfer map $H^n(\Gamma', M) \to H^n(\Gamma, M)$, where Γ' is any torsion-free subgroup of finite index.

Assume now that Γ is a virtual duality group (§3), let $D = H^n(\Gamma, \mathbb{Z}\Gamma)$, and let $\tilde{H}_i(\Gamma, M) = H_i(\Gamma, D \otimes M)$. Then we have:

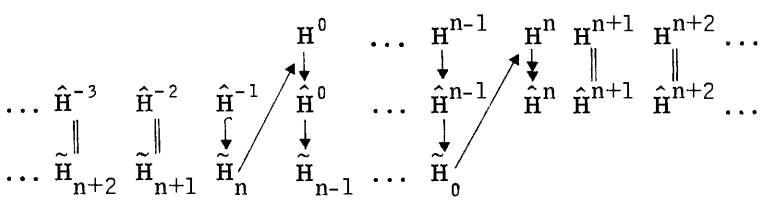
(11.6) $\hat{H}^{-1}(\Gamma)$ is isomorphic to the kernel of the transfer map $\tilde{H}_n(\Gamma) \to \tilde{H}_n(\Gamma')$, with Γ' as in 11.5.

(11.7) $\hat{H}^i = \tilde{H}_{n-i-1}$ for $i < -1$.

(11.8) There is an exact sequence

$$0 \to \hat{H}^{-1} \to \tilde{H}_n \to H^0 \to \hat{H}^0 \to \tilde{H}_{n-1} \to H^1 \to \hat{H}^1 \to \dots \to \tilde{H}_0 \to H^n \to \hat{H}^n \to 0.$$

To summarize, then, the Farrell cohomology theory $\{\hat{H}^i\}$ (at least if Γ is a virtual duality group) consists of the cohomology functors H^i for $i > n$; the homology functors $\tilde{H}_i$ for $i > n$; modified H^n and $\tilde{H}_n$ functors; and n additional functors $\hat{H}^0, \dots, \hat{H}^{n-1}$, which are some sort of mixture of the functors H^i and $\tilde{H}_i$ for $i \leq n$:



Remarks. 1. Properties 11.6-11.8 generalize to the case where Γ is an arbitrary group of type (VFP); the module D must then be replaced by a suitable complex, cf. §3.

2. There are also Farrell homology groups $\hat{H}_*(\Gamma, M)$, defined by

$$\hat{H}_*(\Gamma, M) = H_*(\hat{P} \otimes_{\mathbb{Z}\Gamma} M),$$

and having properties analogous to those above. If Γ is a virtual duality group then one has

$$\hat{H}^i(\Gamma, M) \approx \hat{H}_{n-i-1}(\Gamma, D \otimes M).$$

If, in addition, D is $\mathbb{Z}$-free, then

$$\hat{H}_i(\Gamma, M) \approx \hat{H}^{n-i-1}(\Gamma, \mathrm{Hom}(D, M)).$$

As usual, both of these isomorphisms can be generalized to the case where Γ is only assumed to be of type (VFP).

Finally, we mention that the groups $\hat{H}^*(\Gamma, M)$ are torsion groups; in fact, by transfer theory they are annihilated by the greatest common divisor d of the indices of the torsion-free subgroups Γ' of finite index. One might expect, by analogy with Theorem 7.2, that they are in fact annihilated by the least common multiple m of the orders of the finite subgroups of Γ, but it is not known whether or not this is true. [Note that m and d involve the same primes, and $m \mid d$.] In view of the theory of cup products, it would suffice to show that $1 \in \hat{H}^0(\Gamma, \mathbb{Z})$ is annihilated by m.

Example. Suppose $\Gamma = SL_3(\mathbb{Z})$. Then $d = 48$ and $m = 24$, and the calculations of Soulé [33] show that $\hat{H}^*(\Gamma, \mathbb{Z})$ is indeed annihilated by 24.

§12. Equivariant Farrell cohomology

It has been known for a long time that equivariant cohomology theory provides a machine for relating the cohomology of a discrete group to the cohomology of its finite subgroups. In this section we present the Farrell

cohomology version of this equivariant theory. This generalizes a theory introduced by Swan [37] for finite groups. Throughout this section, Γ denotes an arbitrary group of virtually finite cohomological dimension.

For simplicity, we will define the equivariant cohomology groups $\hat{H}^*_\Gamma(K)$ only in the case where K is an admissible Γ-complex (§8). Moreover, we will assume that K is finite-dimensional. In this case we define, for any Γ-module M,

$$\hat{H}^*_\Gamma(K, M) = H^*(\mathrm{Hom}_{\mathbf{Z}\Gamma}(\hat{P}, C(K, M))).$$

Here $\hat{P}$ is a complete resolution for Γ; $C(K, M)$ is the cellular cochain complex of K with coefficients in the underlying abelian group of M, and $C(K, M)$ is given the diagonal Γ-action; and Hom denotes the total homomorphism complex, i. e. the total complex associated to the double complex $\mathrm{Hom}_{\mathbf{Z}\Gamma}(\hat{P}_\cdot, C^\cdot(K, M))$. As before we will often suppress M and simply write $\hat{H}^*_\Gamma(K)$. Note that $\hat{H}^*_\Gamma(\mathrm{pt.}) = \hat{H}^*(\Gamma)$, hence for any K there is a canonical map

$$\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(K),$$

induced by the map $K \to$ pt. of Γ-complexes.

We immediately obtain from the above definition two spectral sequences converging to $\hat{H}^*_\Gamma(K)$. The first has

$$E_2^{pq} = \hat{H}^p(\Gamma, H^q(K)),$$

and the second has

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} \hat{H}^q(\Gamma_\sigma),$$

where Σ_p is a set of representatives for the p-cells of K mod Γ. The E_2-term of the second spectral sequence is given by

$$E_2^{pq} = H^p(K/\Gamma, \{\hat{H}^q(\Gamma_\sigma)\}),$$

where the right-hand side is to be interpreted as follows. Fix $q \in \mathbf{Z}$. To each cell τ of K/Γ we may associate the group $A_\tau = \hat{H}^q(\Gamma_\sigma)$, where σ is any cell of K lying over τ; this group is independent of the choice of

σ, up to canonical isomorphism. Given a face relation $\tau' < \tau$, we may choose liftings σ' and σ with $\sigma' < \sigma$. We then have $\Gamma_{\sigma'} \supseteq \Gamma_{\sigma}$ by admissibility, hence there is a restriction map $\hat{H}^q(\Gamma_{\sigma'}) \to \hat{H}^q(\Gamma_{\sigma})$ which yields a well-defined map $A_{\tau'} \to A_{\tau}$. These maps satisfy the obvious compatibility condition whenever $\tau'' < \tau' < \tau$, and hence we have a 'coefficient system' on K/Γ. What occurs above, then, is the cohomology of K/Γ with coefficients in this system.

Remarks. 1. A coefficient system of this sort gives rise to a sheaf which is constant with stalk A_{τ} on the interior of τ, and the E_2-term above is isomorphic to the cohomology of K/Γ with coefficients in this sheaf.

2. The first spectral sequence above lives in the first and second quadrants, and the second one lives in the first and fourth quadrants. There is no problem with convergence, however, in view of the finite-dimensionality of K.

We record, now, two properties of equivariant Farrell cohomology which follow easily from the above spectral sequences:

(12.1) If $f : K \to L$ is a cellular Γ-map which induces an isomorphism $H_*K \to H_*L$, then f induces an isomorphism $\hat{H}^*_{\Gamma}(K) \xleftarrow{\cong} \hat{H}^*_{\Gamma}(L)$. In particular, if K is contractible, then $\hat{H}^*_{\Gamma}(K) \approx \hat{H}^*(\Gamma)$.

(12.2) If $K' \subseteq K$ is a Γ-invariant subcomplex such that Γ acts freely in $K - K'$, then $\hat{H}^*_{\Gamma}(K) \xrightarrow{\cong} \hat{H}^*_{\Gamma}(K')$.

Finally, we call attention to an important special case where we will apply the equivariant Farrell theory. Let X be, as in 2.1, a finite-dimensional contractible complex on which Γ acts properly. Then $\hat{H}^*_{\Gamma}(X) \approx \hat{H}^*(\Gamma)$, and the second spectral sequence therefore takes the form:

$$(12.3) \quad E_2^{pq} = H^p(X/\Gamma, \{\hat{H}^q(\Gamma_{\sigma})\}) \Rightarrow \hat{H}^{p+q}(\Gamma).$$

This spectral sequence relates the Farrell cohomology of Γ to the Tate cohomology of its finite subgroups.

(12.4) Exercise. Suppose that X has been chosen so that X^G is connected and non-empty for every finite subgroup $G \subseteq \Gamma$; we know by 2.2 that this is possible. Show that the left-hand edge E_2^{0*} of the above spectral sequence can be identified with $\varprojlim \hat{H}^*(G)$, where G ranges over the finite subgroups of Γ and the limit is taken with respect to all maps between finite subgroups given by conjugation by an element of Γ. Explicitly, an element of this limit is a compatible family $\{u_G\}$, where G ranges over the finite subgroups of Γ, $u_G \in \hat{H}^*(G)$, and the compatibility condition is the following: If G and G' are finite subgroups and γ is an element of Γ such that $\gamma G \gamma^{-1} \subseteq G'$, then $u_{G'}$ maps to u_G under the map $\hat{H}^*(G') \to \hat{H}^*(G)$ induced by conjugation by γ.

§13. Cohomologically trivial modules

Γ continues to denote an arbitrary group of finite virtual cohomological dimension. As an immediate consequence of the equivariant cohomology spectral sequence 12.3, we have:

(13.1) Lemma. *Let M be a Γ-module such that $\hat{H}^*(G, M) = 0$ for every finite subgroup $G \subseteq \Gamma$. Then $\hat{H}^*(\Gamma, M) = 0$.*

We will say that M is *cohomologically trivial* if, as in the lemma, $\hat{H}^*(G, M) = 0$ for every finite $G \subseteq \Gamma$. It then follows from the lemma that $\hat{H}^*(\Gamma_0, M) = 0$ for every subgroup $\Gamma_0 \subseteq \Gamma$.

In case Γ is finite, we have the following characterization of cohomologically trivial modules, due to Rim [27] (see also [29]):

If Γ is finite then a Γ-module M is cohomologically trivial if and only if it has finite projective dimension over $\mathbb{Z}\Gamma$, and in this case $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq 1$. If M is $\mathbb{Z}$-free and cohomologically trivial, then $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M = 0$, i.e. M is $\mathbb{Z}\Gamma$-projective.

We now extend this to the general case. Let $\operatorname{vcd} \Gamma = n$.

(13.2) Theorem. *A Γ-module M is cohomologically trivial if and only if it has finite projective dimension, and in this case $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n + 1$. If M is $\mathbb{Z}$-free and cohomologically trivial then $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n$.*

Proof. Clearly projective modules are cohomologically trivial, hence so is any module of finite projective dimension. Conversely, suppose M is cohomologically trivial, and assume first that M is $\mathbb{Z}$-free. I claim that $\mathrm{Hom}(M, N)$ is cohomologically trivial for any Γ-module N, where $\mathrm{Hom}(\ ,\ ) = \mathrm{Hom}_{\mathbb{Z}}(\ ,\ )$ with the diagonal Γ-action. Indeed, it suffices to verify the claim in case Γ is finite, in which case it follows at once from Rim's theorem. We therefore have $\hat{H}^*(\Gamma, \mathrm{Hom}(M, N)) = 0$, hence

$$\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(M, N) = H^i(\Gamma, \mathrm{Hom}(M, N)) = 0$$

for $i > n$, and $\mathrm{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n$. In case M is not $\mathbb{Z}$-free, choose a surjection $P \to M$ with P projective. The kernel M' of this map will have $\mathrm{proj\,dim}_{\mathbb{Z}\Gamma} M' \leq n$ by what we have just proved, hence $\mathrm{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n + 1$.

§14. Groups with periodic cohomology

A group Γ of finite virtual cohomological dimension will be said to have periodic cohomology if for some integer $d > 0$ there is an element of $\hat{H}^d(\Gamma, \mathbb{Z})$ which is invertible in the ring $\hat{H}^*(\Gamma, \mathbb{Z})$. Cup product with such an element then defines periodicity isomorphisms

$$\hat{H}^i(\Gamma, M) \approx \hat{H}^{i+d}(\Gamma, M)$$

for any Γ-module M and any integer i. Similarly, one can define p-periodicity for a fixed prime p in terms of the existence of an invertible element of positive degree in the ring $\hat{H}^*(\Gamma, \mathbb{Z})_{(p)}$, the p-primary component of $\hat{H}^*(\Gamma, \mathbb{Z})$. [One can show by a Bockstein argument that this is equivalent to the existence of an invertible element of positive degree in $\hat{H}^*(\Gamma, \mathbb{Z}/p)$.] Clearly Γ has periodic cohomology if and only if it has p-periodic cohomology for every prime p.

One way to prove periodicity (or p-periodicity) is by exhibiting a finite quotient Γ/Γ' which has periodic (or p-periodic) cohomology, where Γ' is torsion-free. This follows from the fact that the inflation map (11.2) is a ring homomorphism $\hat{H}^*(\Gamma/\Gamma', \mathbb{Z}) \to \hat{H}^*(\Gamma, \mathbb{Z})$. Similarly,

if Γ has periodic or p-periodic cohomology, then so does every subgroup (because the restriction maps are ring homomorphisms).

The main result on groups with periodic cohomology is the following theorem:

(14.1) Theorem. *Let Γ be a group such that* vcd $\Gamma < \infty$ *and let p be a prime. The following conditions are equivalent:*

(i) *Γ has p-periodic cohomology.*

(ii) *There exist integers i and d with $d > 0$, such that* $\hat{H}^i(\Gamma, M)_{(p)} \approx \hat{H}^{i+d}(\Gamma, M)_{(p)}$ *for all Γ-modules M.*

(iii) *Every finite subgroup of Γ has p-periodic cohomology.*

(iv) *Γ does not contain any subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$.*

(v) *Every finite p-subgroup of Γ is a cyclic or generalized quaternion group.*

Proof. Trivially (i) $\Rightarrow$ (ii). To prove (ii) $\Rightarrow$ (iii), note first that if (ii) holds for some i then it holds for all i by a standard 'dimension shifting' argument. Also, (ii) holds for any subgroup of Γ by Shapiro's lemma. In particular, if $G \subseteq \Gamma$ is finite then $\hat{H}^d(G, \mathbb{Z})_{(p)} \approx \hat{H}^0(G, \mathbb{Z})_{(p)}$, and this is well-known to imply that G has p-periodic cohomology, as required (cf. [16]). The equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are well-known from the theory of finite groups with periodic cohomology [16]; so it remains to prove (iii) $\Rightarrow$ (i).

We recall first that a weaker version of this implication was proved by Venkov ([41], [42]), although he did not use the language of Farrell cohomology theory. (He spoke, rather, of periodicity in the ordinary cohomology of Γ in sufficiently high dimensions.) Restated in terms of Farrell cohomology, his result is the following: If there exists an element $u \in \hat{H}^d(\Gamma, \mathbb{Z})_{(p)}$ $(d > 0)$ whose restriction to $\hat{H}^*(G, \mathbb{Z})_{(p)}$ is invertible for every finite $G \subseteq \Gamma$, then u is invertible and hence Γ has p-periodic cohomology. This result of Venkov is easily deduced from the multiplicative structure in the equivariant cohomology spectral sequence 12.3, localized at p:

$$E_2^{st} = H^s(X/\Gamma, \{\hat{H}^t(\Gamma_\sigma)\})_{(p)} \Rightarrow \hat{H}^{s+t}(\Gamma)_{(p)}.$$

Indeed, one need only observe that multiplication by u induces an isomorphism on E_2 and hence also on the abutment.

To prove (iii) $\Rightarrow$ (i), then, it suffices to prove that (iii) implies the existence of such an element u. This again follows from the multiplicative structure in the above spectral sequence. For let u_G be an invertible element of positive degree in $\hat{H}^*(G, \mathbb{Z})_{(p)}$, where G ranges over the finite subgroups of Γ. Raising u_G to a power, if necessary, we may assume that $\{u_G\}$ is a compatible family in the sense of 12.4, so that $\{u_G\}$ represents an element of the edge E_2^{0*}. It now follows by an argument of Quillen ([25], proof of Prop. 3.2) that some power of $\{u_G\}$ is a permanent cycle in the spectral sequence and hence is the image of some element $u \in \hat{H}^*(\Gamma, \mathbb{Z})$ under the edge homomorphism

$$\hat{H}^*(\Gamma, \mathbb{Z})_{(p)} \to \lim_{\leftarrow} \hat{H}^*(G, \mathbb{Z})_{(p)} . \tag{14.2}$$

This completes the proof.

Remark. In the language of Quillen [25], the above proof is based on the fact that the map 14.2 is an 'F-isomorphism'. It should be noted that Quillen's methods yield the much stronger result (for any group Γ with $\operatorname{vcd} \Gamma < \infty$) that the map

$$\hat{H}^*(\Gamma, \mathbb{Z})_{(p)} \to \lim_{\leftarrow} \hat{H}^*(A, \mathbb{Z})$$

is an F-isomorphism, where A ranges over the elementary abelian p-subgroups of Γ.

§15. The ordered set of finite subgroups

Γ continues to denote an arbitrary group of finite virtual cohomological dimension. As in §8, if Γ operates on a partially ordered set S then we set

$$\hat{H}^*_\Gamma(S) = \hat{H}^*_\Gamma(K(S)) .$$

As usual it is understood here that there is an arbitrary Γ-module M of coefficients. In this section we will prove analogues in Farrell cohomology

of Theorems 8.3 and 8.4.

Recall that, for any finite-dimensional admissible Γ-complex K, there is a canonical map $\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(K)$.

(15.1) Theorem. Let $\mathcal{F}$ be the set of non-trivial finite subgroups of Γ. Then the canonical map

$$\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(\mathcal{F})$$

is an isomorphism.

Proof. Let X and X_0 be as in §9, proof of Theorem 8.3. Then

$$\begin{aligned} \hat{H}^*(\Gamma) &\approx \hat{H}^*_\Gamma(X) && \text{by 12.1} \\ &\approx \hat{H}^*_\Gamma(X_0) && \text{by 12.2} \\ &\approx \hat{H}^*_\Gamma(\mathcal{F}) && \text{by 12.1 and 9.5.} \end{aligned}$$

It is easy to check that this composite isomorphism is in fact given by the canonical map $\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(\mathcal{F})$.

Next we prove the analogue of 9.6:

(15.2) Proposition. Let K be a finite-dimensional admissible Γ-complex and let p be a prime such that K^H is contractible for every non-trivial p-subgroup $H \subseteq \Gamma$. Then

$$\hat{H}^*(\Gamma)_{(p)} \xrightarrow{\approx} \hat{H}^*_\Gamma(K)_{(p)} .$$

Proof[2]. Let $\Gamma' \subseteq \Gamma_p \subseteq \Gamma$ be as in the proof of 7.1. Since $(\Gamma : \Gamma_p)$ is relatively prime to p, we may use restriction and transfer maps in the usual way to obtain, for any finite-dimensional admissible Γ-complex L, a natural embedding of $\hat{H}^*_\Gamma(L)_{(p)}$ as a direct summand of $\hat{H}^*_{\Gamma_p}(L)$. Applying this to $L = K$ and $L = \text{pt.}$, we see that the canonical map $\hat{H}^*(\Gamma)_{(p)} \to \hat{H}^*_\Gamma(K)_{(p)}$ is a direct summand of the canonical map $\hat{H}^*(\Gamma_p) \to \hat{H}^*_{\Gamma_p}(K)$. It therefore suffices to prove that the latter is an

[2] I am grateful to D. Quillen for a suggestion which simplified my original proof.

isomorphism. Just as in the proof of 9.6, we now apply the method of proof of Theorem 15.1 to $(\Gamma_p, X \times K)$ and (Γ_p, X) to deduce

$$\hat{H}^*_{\Gamma_p}(K) \approx \hat{H}^*_{\Gamma_p}(\mathcal{F}) \approx \hat{H}^*(\Gamma_p),$$

where $\mathcal{F}$ is now the set of non-trivial finite subgroups of Γ_p. It is easily checked that the composite isomorphism is given by the canonical map, whence the proposition.

Let $\mathcal{A}_p$ be the set of non-trivial elementary abelian p-subgroups of Γ. As in §9 (proof of Theorem 8.4), we may apply 15.2 with $K = K(\mathcal{A}_p)$ to obtain:

(15.3) Theorem. For any prime p,

$$\hat{H}^*(\Gamma)_{(p)} \xrightarrow{\approx} \hat{H}^*_\Gamma(\mathcal{A}_p)_{(p)} .$$

This theorem gives information about $\hat{H}^*(\Gamma)_{(p)}$ in terms of the elementary abelian p-subgroups and their normalizers, cf. [14], Prop. 2. In case Γ contains no elementary abelian p-group of rank 2 (i.e. if Γ has p-periodic cohomology), this information takes the following simple form:

(15.4) Corollary. If Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$, then

$$\hat{H}^*(\Gamma)_{(p)} \approx \Pi\, \hat{H}^*(N(P))_{(p)} ,$$

where P ranges over the subgroups of Γ of order p, up to conjugacy.

In [14], §6, we applied the corollary with $\Gamma = SL_3(\mathbb{Z})$ to calculate the 3-primary component of $\hat{H}^*(SL_3(\mathbb{Z}))$, from which we obtained $H^*(SL_3(\mathbb{Z}), \mathbb{Z})$ and $H_*(SL_3(\mathbb{Z}), St)$ modulo 2-torsion. Here St denotes the Steinberg module, cf. 3.3. (Of course, these calculations have been subsumed in the work of Soulé [33].) We now give another example in which Theorem 15.3 yields concrete information relating $\hat{H}^*(\Gamma)$ to the cohomology of the normalizers of the elementary abelian p-subgroups of Γ.

Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}])$. We will apply Theorem 15.3 with $p = 2$. Note first that every elementary abelian 2-subgroup of Γ is diagonalizable and hence has rank ≤ 2; thus $K(\mathcal{A}_2)$ is a graph. Let P_0 be the group of

order 2 generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let P_1 be the group of order 4 consisting of the matrices

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

in SL_3. Then $P_0 \subset P_1$ so there is an edge of $K(\mathfrak{A}_2)$ with vertices P_0 and P_1, and it is easy to see that this edge is a fundamental domain for the action of Γ on $K(\mathfrak{A}_2)$. The isotropy groups of the vertices P_0 and P_1 are the normalizers $N(P_0)$ and $N(P_1)$. Explicitly, $N(P_0)$ is isomorphic to $GL_2 (= GL_2(\mathbb{Z}[\frac{1}{2}]))$, embedded in SL_3 in the usual way,

$$A \mapsto \left(\begin{array}{cc|c} & & 0 \\ & A & 0 \\ \hline 0 & 0 & \det A^{-1} \end{array}\right),$$

and $N(P_1)$ is the group SM_3 of monomial matrices in SL_3. The isotropy group of the edge (P_0, P_1) is $GL_2 \cap SM_3 = M_2$, the group of 2×2 monomial matrices. The second spectral sequence of equivariant cohomology theory (§12) therefore yields a 'Mayer-Vietoris' sequence relating $\hat{H}^*_\Gamma(\mathfrak{A}_2)$ to $\hat{H}^*(GL_2)$, $\hat{H}^*(SM_3)$, and $\hat{H}^*(M_2)$, so we have by Theorem 15.3 a Mayer-Vietoris sequence

$$\ldots \to \hat{H}^{i-1}(M_2)_{(2)} \to \hat{H}^i(SL_3)_{(2)} \to \hat{H}^i(GL_2)_{(2)} \oplus \hat{H}^i(SM_3)_{(2)} \to \hat{H}^i(M_2)_{(2)} \to \ldots .$$

The result can be stated more precisely, as follows. Let $\tilde{\Gamma} = GL_2 *_{M_2} SM_3$ and consider the canonical map $\tilde{\Gamma} \to \Gamma$. One can show that $K(\mathfrak{A}_2)$ is connected, whence this map is surjective and its kernel is isomorphic to the free group $\pi_1(K(\mathfrak{A}_2))$, cf. [31], ch. I, 5.4, Ex. 3. There is therefore an inflation map (11.2)

$$\hat{H}^*(\Gamma) \to \hat{H}^*(\tilde{\Gamma}),$$

and our result is that this induces an isomorphism on 2-primary components, with any Γ-module of coefficients. In particular, since vcd $\Gamma = 5$ by Borel-Serre [9], we have

$$H^i(\Gamma)_{(2)} \xrightarrow{\approx} H^i(\tilde{\Gamma})_{(2)}$$

for $i > 5$.

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3 · Free abelianised extensions of finite groups

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The central subject matter of these notes is the class of groups of the form $F/[R, R]$, where F is a finitely generated free group and F/R is isomorphic to a given finite group G. Lecture 1 deals with the relations between different such covering groups of a fixed group G; Lecture 2 with their decompositions; and Lecture 3 with their generation properties.

The lectures constitute a report on the present state of knowledge concerning these topics. I have tried to explain fully the various concepts that arise and the connexions between them, but I have had to omit almost all proofs. Nevertheless, I hope that this account will be found accessible by the reader who is interested in presentations of groups but does not have the rather specialised background in module theory necessary for many proofs. This background might be called 'the K_0-theory of finite groups'. Perhaps the best reference for it is Swan's volume in the Springer Lecture Notes series, no. 149 ('K-theory of finite groups and orders').

I: FREE EXTENSIONS AND THE COMPARISON PROBLEM

§1 Introduction

Problems connected with presentations of groups arise within group theory in two essentially different contexts. In the first of these, we are given a group by means of generators and relations and wish to deduce structural information about the group. The theory of free products and the theory of groups with a single defining relation are typical examples of this situation. In the other context, the group is supposed to be known structurally or representation theoretically (but possibly in quite a weak sense) and the aim is to deduce extension theoretic consequences. Questions concerning finite presentability fall naturally under this heading. In

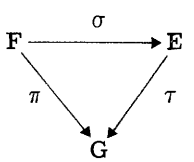
these lectures we shall be concerned only with the second of these contexts.

Explicitly, suppose G is our given group and E is a group in some sensible class all of whose members have G as homomorphic image. Thus there exist homomorphisms of E onto G and the aim is to study these and their kernels.

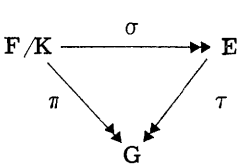
We look briefly at an important example. Let $\tau : E \twoheadrightarrow G$ and suppose (i) $N = \operatorname{Ker} \tau$ is central in E: i.e. $[N, E] = 1$; and (ii) that $N \leq E' = [E, E]$ (which implies that N has no supplement in E and thus G does not arise from a group similar to E but 'smaller' than E: if $E_1 N = E$, then $E_1' = E'$, whence $E_1 = E$). Take any free presentation of G:

$$R \rightarrowtail F \overset{\pi}{\twoheadrightarrow} G$$

and lift π to a homomorphism σ:



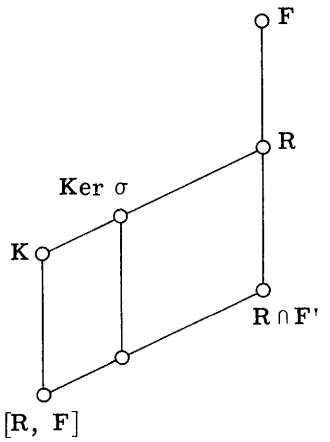
Then σ is necessarily surjective (hypothesis (ii) on N) and $[R, F]\sigma = 1$. Moreover, $\operatorname{Ker} \sigma$ contains a complement K to $R \cap F'$ modulo $[R, F]$. (For if $L = \operatorname{Ker} \sigma$, then $N \leq E'$ implies $R \leq F'L$ and hence $R = (R \cap F')L$. Thus $L/L \cap R \cap F'$ is free abelian and therefore $L/[R, F]$ splits over $L \cap R \cap F'/[R, F]$. A complement is then automatically a complement to $R \cap F'/[R, F]$ in $R/[R, F]$.) Hence σ actually induces an epimorphism



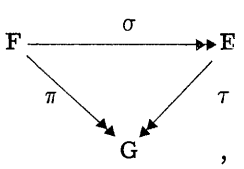
and R/K is mapped by σ onto N. Now

$$R/K \cong R \cap F'/[R, F] \cong H_2(G, \mathbb{Z})$$

and hence is an invariant of G. It follows that the size of N is limited. The groups F/K are the Schur covering groups of G (called Darstellungs gruppen by Schur).



A less restrictive version of the above situation arises when N (i) is merely assumed to be abelian ($N' = 1$) and (ii) possesses no supplement in E. Here $\pi : F \twoheadrightarrow G$ lifts as before to an epimorphism σ:



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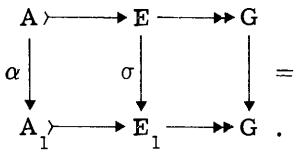
but now all we know is that $R'\sigma = 1$: i.e. σ induces $F/R' \twoheadrightarrow E$ and R/R' maps onto N.

The free abelian group R/R' has the structure of a G-module via conjugation by F and is called the relation module determined by the free presentation π.

The significance of the extension F/R' is best expressed in a categorical form. We consider the class of all extensions by G with abelian kernel and make these the objects of a category $(\underline{\underline{G}})$. The morphisms are the homomorphisms of extensions that induce the identity on G:



We shall abbreviate the above diagram as $(\alpha|\sigma)\colon (A|E) \to (A_1|E_1)$. Free objects can now be defined via universality in a sensible manner and it turns out (cf. [7], §9. 5) that the free objects are precisely the extensions

$$(\bar{R}|\bar{F})\colon \bar{R} \rightarrowtail \bar{F} \twoheadrightarrow G,$$

where we have written $\bar{R} = R/R'$, $\bar{F} = F/R'$ (and F is a free group).

Similarly, if $(\frac{G}{Tr})$ is the category of all extensions by G with central kernel (cf. [7], §9. 9), the free objects are all those of the form

$$R/[R,\ F] \rightarrowtail F/[R,\ F] \twoheadrightarrow G.$$

The free extensions $(\bar{R}|\bar{F})$ form the main object of study in these lectures. Perhaps the first point to make is that these extensions can actually provide information about arbitrary extensions $N \rightarrowtail E \twoheadrightarrow G$, where N is not necessarily abelian. As an important illustration, consider the problem of the minimum number of defining relations.

First some notation (for use here and later). If X is a group and P is a group of operators on X, then we shall write $d_P(X)$ for the minimum number of elements needed to generate X as P-group. The three basic situations of this type are (i) X is a normal subgroup of P and P acts by conjugation; (ii) X is a P-module; (iii) $P = 1$, the trivial group, and here we shall write $d(X) = d_1(X)$ (which is then the minimum number of group generators of X).

We return now to our extension $N \rightarrowtail E \twoheadrightarrow G$. It is not difficult to show that if E is finite and $N > N'$, then $d_E(N) = d_E(N/N')$ (cf. [8], p. 9). Much more generally, Akbar Rhemtulla has proved that if $d_E(N)$ is finite and $N > N'$, then $d_E(N) = d_E(N/N')$ provided N has the following property:

(*) there does not exist an infinite descending series of E-normal subgroups $N = C_0 > C_1 \ldots$ with each C_i/C_{i+1} perfect.

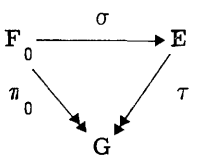
It follows that if $(\bar{R}|\bar{F})$ is a free extension in $(\frac{G}{-})$ with $d_F(R)$ finite, then $d_G(\bar{R})$ is an attained upper bound for the set of all $d_F(R/K)$, where R/K has the property (*) of N above (with $E = F/K$).

As far as I know, no case is known where $d_F(R) > d_G(\bar{R})$. The above result underscores the difficulty of constructing such a presentation.

We shall return to this problem at the end of the last lecture.

§2. Nielsen equivalence

Suppose we are given a free presentation $R_0 \rightarrowtail F_0 \overset{\pi_0}{\twoheadrightarrow} G$ and also an extension $N \rightarrowtail E \overset{\tau}{\twoheadrightarrow} G$. We can lift π_0 to σ:



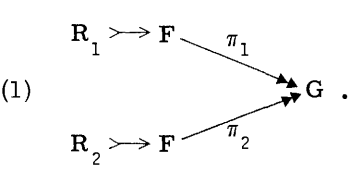
but, in the absence of any supplementation condition on N, σ need not be surjective. (For example, if $d(F_0) = d(G) < d(E)$, then σ cannot be an epimorphism.)

However, starting from any free presentation of E: $\sigma_1 : F_1 \overset{\pi_1}{\twoheadrightarrow} E$, we can obtain a free presentation of G, viz. $R_1 \rightarrowtail F_1 \overset{\pi_1}{\twoheadrightarrow} G$, where $\pi_1 = \sigma_1\tau$.

Suppose $d(F_0) \leq d(F_1)$ and let $F_2 = F_0 * L \cong F_1$. Then we obtain a new free presentation of G by

$$\pi_2 = \pi_0 * (\text{collapse}): \quad R_2 \rightarrowtail F_2 \twoheadrightarrow G.$$

Identifying F_1 and F_2 gives two free presentations of G from the same free group F:



(1)

How do they compare?

The nicest possible connexion arises if there exists an automorphism α of F so that $\alpha\pi_2 = \pi_1$. We then say that π_1 and π_2 are Nielsen equivalent. If Epi(F, G) denotes all epimorphisms of F onto G, then $\pi \to \alpha^{-1}\pi$ defines a permutational representation of Aut F on Epi(F, G) and the orbits are precisely the Nielsen equivalence classes for G over F.

Rather little is known about this permutational representation of Aut F. We state here two facts for a finite group G (and these are about the only reasonably general results available at present):

(2) Let G be a finite group. Then Aut F acts transitively on Epi(F, G) (i.e. any two free presentations from F are Nielsen equivalent) if

(i) $d(F) \geq 2 \log_2 |G|$ (Gilman [6]),

(ii) G is soluble and $d(F) \geq 1 + d(G)$ (Dunwoody [4]).

The enormous gap between these two results suggests a whole string of questions. For example: is there some sensible (group theoretic) description of the class of finite groups G for which Aut F operates transitively on Epi(F, G), where $d(F) = 1 + d(G)$? (There is some interesting information on this point in the paper of Gilman referred to above. Cf. also [3].) If $k = k(G)$ is the smallest integer so that Aut F operates transitively on Epi(F, G), where $d(F) = k$, what structural information of G does k depend on?

The approach to presentation-theoretic problems involving Nielsen classes goes back to work of Hall and Neumann in the mid-thirties ([13], [16]). Hall works with the concept of G-defining subgroup of F. This means a subgroup which is the kernel of an element of Epi(F, G). If we allow Aut G to operate on Epi(F, G) via $\pi \to \pi\sigma$ ($\sigma \in$ Aut G), then the set of orbits Epi(F, G)/Aut G is naturally bijective with the set of all G-defining subgroups of F. Observe that we now have an action of Aut F $\times$ Aut G on Epi(F, G). The orbits under this group are Neumann's T-systems (for G over F).

§3. Comparing free extensions

If π_1, π_2 (as in diagram (1) above) are Nielsen equivalent, then obviously $(\overline{R}_1 | \overline{F})$, $(\overline{R}_2 | \overline{F})$ are isomorphic free extensions in $(\frac{G}{-})$. But if we only require this conclusion, then we can do considerably better than Theorem (2). To state the result, which is due to Peter Linnell (unpublished) and is based on earlier work of J. S. Williams [25], it is convenient to have the following definition.

The generation gap of G is the non-negative integer

$$\text{gap}(G) = d(G) - d_G(\mathfrak{g}),$$

where $\mathfrak{g}$ is the augmentation ideal of G (i.e. the kernel of the ring homomorphism $\mathbb{Z}G \to \mathbb{Z}$ determined by $g \mapsto 1$, $g \in G$). Note that gap(G) is defined for any finitely generated group.

(3) *If G is finite and $d(F) \geq \max\{d(G),\ d(G) + 1 - \text{gap}(G)\}$, then any two free extensions from F are isomorphic:* $(\overline{R}_1 | \overline{F}) \cong (\overline{R}_2 | \overline{F})$.

We shall make a brief comment about the proof of this theorem later (§5). Note that, in particular, the conclusion holds for every finite group G if $d(F) \geq 1 + d(G)$. This makes it very likely that Nielsen equivalence is genuinely more restrictive than the isomorphism of free extensions. But I believe no explicit example to this effect has appeared in the literature.

The gap function will be discussed in more detail in Lecture 3. For the present we remark only that every finite soluble group G has $\text{gap}(G) = 0$ and that there do exist groups whose gap is any given positive integer.

The conclusion of Theorem (3) can easily fail when $d(F) = d(G)$. For if A is a finite abelian group and $d(F) = d(A)$, then the set of isomorphism classes of free extensions $(\overline{R} | \overline{F})$ is bijective with the Nielsen classes over F (Williams [25]). The number of these Nielsen classes has been calculated by Dunwoody in his thesis: if n is the smallest invariant factor (torsion coefficient) of A, then the number is

$\frac{1}{2}\phi(n)$ if $n \neq 2$

1 if $n = 2$.

(Here ϕ is Euler's function.)

The minimal free extensions of A cannot, in fact, be wildly different: for there is here only one T-system. This elementary fact should be compared with Dunwoody's construction [3], for each prime p and positive integer m, of a p-group G of class 2 having at least m T-systems (necessarily over a free group of the minimum possible rank d(G)). (The number d(G) can also be specified initially.)

Despite all this, the free extensions arising from a single free group F are always closely related to one another, even when $d(F) = d(G)$. In fact they lie in a single genus class. The meaning of this concept is easy to explain once we have constructed a translating device from group extensions to module theory (§5).

§4. Comparing relation modules

If $(\overline{R}_1 | \overline{F}) \cong (\overline{R}_2 | \overline{F})$, then obviously $\overline{R}_1 \cong \overline{R}_2$. That the converse is not always true follows from results already mentioned in the last section. If G is cyclic of prime order $p \geq 5$, then there are $\frac{1}{2}(p - 1)$ non-isomorphic minimal free extensions but, of course, the trivial module $\mathbb{Z}$ is the only minimal relation module.

We are here using 'minimal' in the expected sense: for any finitely generated group G, a free presentation $R \rightarrowtail F \twoheadrightarrow G$ is called minimal if $d(F) = d(G)$. Then the corresponding extension $(\overline{R} | \overline{F})$ is a minimal free extension and $\overline{R}$ is a minimal relation module.

For the rest of this section let us assume our group G is finite.

It follows from (3) that if $\overline{R}_1$, $\overline{R}_2$ are non-minimal relation modules arising from the same free group F, then $\overline{R}_1 \cong \overline{R}_2$. Further, if $\overline{R}_1$, $\overline{R}_2$ are minimal and $\mathrm{gap}(G) > 0$, then we still have $\overline{R}_1 \cong \overline{R}_2$.

However, the non-vanishing of the generation gap is by no means a necessary condition for minimal relation modules to be isomorphic. Williams [25] observed that whenever the Jacobinski cancellation theorem [14] can be applied (and the possibility of this depends on the construction

in §5), then two minimal relation modules are isomorphic. Explicitly, this gives

(4) The minimal relation modules $\bar{R}_1$, $\bar{R}_2$ are isomorphic if (i) $d(G) \geq 3$ or $\mathbb{Z}G$ satisfies the Eichler condition and (ii) $\mathbb{Z}G$ is a direct summand of the direct sum of n copies of $\bar{R}_1$, for some n.

The restriction implied by condition (i) is not serious: the Eichler condition holds if G does not have as homomorphic image a sub-group of the non-zero real quaternions that spans the quaternions over $\mathbb{R}$; and this rules out a small family of explicitly known groups ([28], §2.6). Condition (ii), however, is troublesome. Williams [27] has recently verified this for the groups A_n, S_n $(n \geq 5)$, $PSL(n, q)$ $(q \neq 2$ and $(n, q) \neq (2, 3))$. These groups all satisfy condition (i) and hence have a unique isomorphism class of minimal relation modules. None of these groups are covered by (3) since they all have gap 0 (because every 2-generator group has gap 0: e.g. [9] p. 217).

These facts raise the possibility that perhaps minimal relation modules are always isomorphic. But this has recently been shown to be not so by Dyer and Sieradski [18]. They produce a lower bound, which is often greater than 1, for the number of isomorphism classes of minimal relation modules of any (finite) group G satisfying $d(G) = d(G/G')$. It may be relevant to remark that this condition on $d(G)$ implies $gap(G) = 0$ (cf. Lecture 3 or [9]).

The methods of Dyer and Sieradski have been refined by Peter Webb, who calculates the exact number of isomorphism classes of minimal relation modules when G is abelian. We describe his result. Take $\alpha : A \twoheadrightarrow G$ to be a presentation with A free abelian of rank $d(A) = d(G)$. Let Ω be the subgroup of $\mathrm{Aut}\, G$ consisting of all those automorphisms induced via α by automorphisms of A. (This subgroup does not depend on α.) For each integer k prime to $|G|$, write m_k for the automorphism $g \mapsto g^k$. By work of Swan [20], the ideal

$$I(k) = k\,\mathbb{Z}G + \Big(\sum_{g \in G} g\Big)\mathbb{Z}$$

is projective; and if S is the set of all m_k so that $I(k)$ is a free module,

then S is a subgroup of Aut G. Webb's formula states that the number of isomorphism classes of minimal relation modules of G is $(\mathrm{Aut}\, G : \Omega S)$.

§5. A link to module theory

Recall that the augmentation ideal $\mathfrak{g}$ of G (strictly of $\mathbb{Z}G$) is the kernel of the map $\mathbb{Z}G \to \mathbb{Z}$ defined by taking coefficient sums:

$$\sum m_i g_i \to \sum m_i .$$

This is a two-sided ideal of $\mathbb{Z}G$.

We adopt the convention that a small german letter shall denote the augmentation ideal of the group given by the corresponding capital latin letter.

Let $N \rightarrowtail E \overset{\tau}{\twoheadrightarrow} G$ be an extension. Then τ extends by $\mathbb{Z}$-linearity to a homomorphism $\tau : \mathbb{Z}E \to \mathbb{Z}G$, whose kernel is νE, the right ideal (and therefore two-sided ideal) generated by ν. Restricting τ to the augmentation ideal $\mathfrak{e}$ of E gives

$$(5) \qquad 0 \to \nu E/\mathfrak{e}\nu \to \mathfrak{e}/\mathfrak{e}\nu \to \mathfrak{g} \to 0,$$

which is plainly an exact sequence of G-modules. It is easily proved (cf. [8], p. 6) that

$$(6) \qquad \nu E/\mathfrak{e}\nu \cong N/N',$$

as G-modules. (The isomorphism is the obvious one: $(n-1)+\mathfrak{e}\nu \mapsto nN'$, $n \in N$.)

We are now prompted to introduce a new category $(\underline{\mathfrak{g}})$, whose objects shall be the G-module extensions

$$(A\,|\,M) : A \rightarrowtail M \twoheadrightarrow \mathfrak{g}$$

and morphisms shall be diagrams of the following type:

$$\begin{array}{ccc} (A\,|\,M) & A & \rightarrowtail M \twoheadrightarrow \mathfrak{g} \\ (\alpha\,|\,\mu)\ \downarrow & \alpha\downarrow & \mu\downarrow \quad\ \downarrow = \\ (A_1\,|\,M_1) & A_1 & \rightarrowtail M_1 \twoheadrightarrow \mathfrak{g} \end{array} .$$

If $A \rightarrowtail E \twoheadrightarrow G$ is an object in $(\underline{G})$, we use (5) and (6) above to construct the object

$$A \rightarrowtail e/e\alpha \twoheadrightarrow g \quad .$$

in $(\underline{g})$. This yields a functor $\Phi : (\underline{G}) \rightarrow (\underline{g})$.

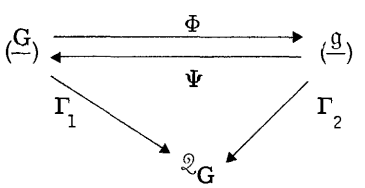
There is also a functor in the opposite direction. Given $A \rightarrowtail M \overset{\phi_0}{\twoheadrightarrow} g$, we can form the group extension

$$A \rightarrowtail G \wr M \overset{\phi}{\twoheadrightarrow} G \wr g,$$

where $G \wr V$ denotes the split extension of a G-module V by G. The homomorphism ϕ is given by $(g, m) \rightarrow (g, m\phi_0)$. There exists a homomorphism $\psi : G \rightarrow G \wr g$ defined by $g \mapsto (g, g - 1)$. If $E = G\psi\phi^{-1}$, then $A \rightarrowtail E \overset{\phi}{\twoheadrightarrow} G$ is an object in $(\underline{G})$. We now have a functor $\Psi : (\underline{g}) \rightarrow (\underline{G})$.

(7) The functors Φ, Ψ determine a natural equivalence between the categories $(\underline{g})$ and $(\underline{G})$.

Let $\mathfrak{L}_G$ denote the category of pairs (A, x), where A is a G-module and $x \in H^2(G, A)$. A morphism $\alpha : (A, x) \rightarrow (A_1, x_1)$ is a G-module homomorphism $\alpha : A \rightarrow A_1$ so that $xH^2(G, \alpha) = x_1$. The classical extension theory of groups provides a functor $\Gamma_1 : (\underline{G}) \rightarrow \mathfrak{L}_G$, which is surjective (full and representative). Now $H^2(G, \ ) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(g, \ )$ and the extension theory of modules then gives a surjective functor $\Gamma_2 : (\underline{g}) \rightarrow \mathfrak{L}_G$. Moreover the following two triangles commute:



Since a certain amount of group theoretic and module theoretic information is lost by applying Γ_1 and Γ_2, respectively, it follows that the direct link provided by (7) is better than the indirect one via $\mathfrak{L}_G$ (apart from being considerably easier). (For details and further information cf. [7], §9.1 and [12]. My attention was recently drawn to an early paper

on these matters by R. H. Crowell [2].)

We may use (7) to translate extension theoretic concepts, problems and arguments from G to $\mathfrak{g}$ and vice versa. Of course, the dictionary can be used only for constructs that are invariant under a natural equivalence of categories.

We promised a comment about the proof of (3) (§3). If $(\bar{R}_1 | \bar{F})$, $(\bar{R}_2 | \bar{F})$ are two free extensions, then (5) and (6) yield two presentations of $\mathfrak{g}$:

$$(8) \qquad \bar{R}_i \rightarrowtail \mathfrak{f} / \mathfrak{f} r_i \twoheadrightarrow \mathfrak{g}, \qquad i = 1, 2.$$

These are actually free module presentations of $\mathfrak{g}$: for $\mathfrak{f}$ is a free module over $\mathbb{Z}F$, freely generated by $\{x_i - 1, i \in I\}$, where $\{x_i, i \in I\}$ is a set of free generators of F; whence $\mathfrak{f}/\mathfrak{f} r_i$ is G-free. Schanuel's lemma now shows that

$$\bar{R}_1 \oplus (\mathbb{Z}G)^{d(F)} \cong \bar{R}_2 \oplus (\mathbb{Z}G)^{d(F)}.$$

Under suitable hypotheses (such as (4)(i)), we can cancel and obtain $\bar{R}_1 \cong \bar{R}_2$. The basic point of the proof of (3) is to show that a given isomorphism between $\bar{R}_1$ and $\bar{R}_2$ can be adjusted to become an isomorphism of the module extensions (8).

A second outstanding point concerns the question of genus. Let G be finite and recall that a $\mathbb{Z}G$-module A is called a lattice if the $\mathbb{Z}$-module structure of A is that of a finitely generated free $\mathbb{Z}$-module. Two $\mathbb{Z}G$-lattices A, B are in the same genus if they are locally isomorphic: i.e. $A \otimes \mathbb{Z}_{(p)} \cong B \otimes \mathbb{Z}_{(p)}$ for all primes p ($\mathbb{Z}_{(p)}$ being the local ring at p). Given two extensions $(A_1 | M_1)$, $(A_2 | M_2)$ in $(\mathfrak{G})$, where A_1, A_2 are $\mathbb{Z}G$-lattices, then automatically M_1, M_2 are also lattices (because $\mathfrak{g}$ is one). We say $(A_1 | M_1)$, $(A_2 | M_2)$ are in the same genus if, for each prime p, there exists an isomorphism of extensions:

$$\begin{array}{ccccc}
A_1 \otimes \mathbb{Z}_{(p)} & \rightarrowtail & M_1 \otimes \mathbb{Z}_{(p)} & \twoheadrightarrow & \mathfrak{g} \otimes \mathbb{Z}_{(p)} \\
\downarrow & & \downarrow & & \| \\
A_2 \otimes \mathbb{Z}_{(p)} & \rightarrowtail & M_2 \otimes \mathbb{Z}_{(p)} & \twoheadrightarrow & \mathfrak{g} \otimes \mathbb{Z}_{(p)}
\end{array}$$

Two extensions in $(\underline{\underline{G}})$ with lattice kernels, are then said to be in the same genus if their images in $(\underline{\underline{\mathfrak{g}}})$ are in the same genus. (Using (7), this can be expressed purely group theoretically.) We have now attached a meaning to the assertion in the last paragraph of §3.

An example of a concept that does not translate satisfactorily is that of freeness. An extension $(A|M)$ is free in $(\underline{\underline{\mathfrak{g}}})$ if, and only if, M is a free $\mathbb{Z}G$-module. The image of a free extension $(\overline{R}|\overline{F})$ in $(\underline{\underline{G}})$ is indeed a free extension $(\overline{R}|f/fr)$ in $(\underline{\underline{\mathfrak{g}}})$, but of a rather special sort: f/fr has a set of free generators $\{x_i, i \in I\}$ so that $\{x_i\pi + 1\}$ is a generating set of G (where π is $f/fr \twoheadrightarrow g$). A. J. McIsaac has proved that if G is finite abelian, then every free extension in $(\underline{\underline{\mathfrak{g}}})$ does indeed have this additional property and therefore, in this case, all free objects in $(\underline{\underline{\mathfrak{g}}})$ do have free images in $(\underline{\underline{G}})$. It would be of considerable interest to know for what other classes of groups this assertion is true. It is definitely false for all groups of positive generation gap.

II: PROJECTIVE EXTENSIONS AND THE DECOMPOSITION PROBLEM

§6. The decomposition problem

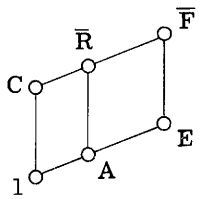
In the last lecture we were primarily interested in the free extensions $(\overline{R}|\overline{F})$ in $(\underline{\underline{G}})$. We turn now to the projective extensions (projective objects) These are defined in the usual way by the completion property for triangles: $(A|E)$ is projective if, and only if, for every diagram

$$\begin{array}{ccc} & & (A|E) \\ & & \downarrow (\alpha|\sigma) \\ (A_1|E_1) & \xrightarrow{(\alpha'|\sigma')} & (A_2|E_2) \end{array}$$

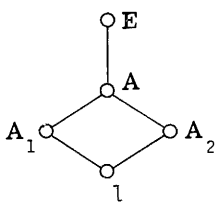
with $(\alpha'|\sigma')$ an epimorphism, there exists a morphism $(\alpha''|\sigma''):(A|E) \to (A_1|E_1)$ so that $(\alpha''|\sigma'')(\alpha'|\sigma') = (\alpha|\sigma)$. (The morphism $(\alpha'|\sigma')$ is an epimorphism if, and only if, α' (or equivalently σ') is surjective.) (Cf. [7], chapter 9 and [11] for details on these and the following matters.)

It is easy enough to describe the projectives. In $(\underline{\underline{\mathfrak{g}}})$, the extension $(A|P)$ is projective if, and only if, P is a projective $\mathbb{Z}G$-module. In $(\underline{\underline{G}})$, the extension $(A|E)$ is projective if, and only if, there is a free

extension $(\bar{R}\,|\,\bar{F})$, a module decomposition $\bar{R} = A \oplus C$ so that $\bar{F}$ splits over C and then $(\bar{R}/C\,|\,\bar{F}/C) \cong (A\,|\,E)$. It follows automatically that C is a projective $\mathbb{Z}G$-module.



A nicer way of expressing this is to use the observation that $(\underline{\underline{G}})$ has finite products. In fact, if $(A\,|\,E)$ is a given extension and $A = A_1 \oplus A_2$, then $(A\,|\,E) \cong (A/A_1\,|\,E/A_1) \amalg (A/A_2\,|\,E/A_2)$.



Thus we can state

(9) The group extension $(A\,|\,E)$ is projective if, and only if, there exists a $\mathbb{Z}G$-module C so that

$$(A\,|\,E) \amalg (C\,|\,C \rtimes G)$$

is a free extension. Moreover, C is then a $\mathbb{Z}G$-projective module.

The problem we propose to study is the following: Is every factorisation of a free extension necessarily of the form (9)? Explicitly, if

$$(\bar{R}\,|\,\bar{F}) \cong (A_1\,|\,E_1) \amalg (A_2\,|\,E_2)\,,$$

does one factor have to be a split extension with ZG-projective kernel? In view of (9), this is a necessary and sufficient condition for the other factor to be a projective object in $(\underline{\underline{G}})$.

We can rephrase the problem in terms of the pair category $\mathfrak{L}_G$ introduced in §5. If χ is the cohomology class of the extension $(\bar{R}\,|\,\bar{F})$ and

$(\bar{R}, \chi) \cong (A_1, x_1) \amalg (A_2, x_2)$

(so that $\bar{R} \cong A_1 \oplus A_2$ and x_i is the image of χ under the map induced by the projection $\bar{R} \twoheadrightarrow A_i$), does it follow that one of x_1, x_2 is 0?

For the rest of this lecture we shall assume G is a finite group and that all extensions are finitely generated. Since $H^*(G, \text{projective})=0$, our problem is now equivalent to the following:

(10) *If $\bar{R}$ is a relation module and $\bar{R} = A_1 \oplus A_2$, must then one of A_1, A_2 be projective?*

Our method of tackling this question proceeds by way of an important class of lattices, which we call Heller modules (§7) and which enable one to study decomposability problems in an integral setting. The Heller property for $\mathbb{Z}$ leads us to the prime graph of a group (§8), an object that is often of interest in abstract group theory. In this way we shall obtain (in §10) an internal structural condition for answering (10). In the final lecture we shall find that these results also provide indecomposability criteria for minimal relation modules.

§7. Heller modules

The first observation to be made about (10) is that this is really a problem only about the genus class of $\bar{R}$. For let us call a property of $\mathbb{Z}G$-lattices a *genus property* if, whenever a particular lattice has the property, then all lattices in the same genus also have it. Then the property of being projective and that of being indecomposable are both genus properties; and so is the property of admitting a non-projective decomposition: $L = L_1 \oplus L_2$ is a non-projective decomposition if neither L_1 nor L_2 is projective. (Cf., e.g., [8], §5.1.)

Suppose $\bar{R} = A_1 \oplus A_2$ is a non-projective decomposition of the relation module $\bar{R}$. We show that the augmentation ideal g 'stably decomposes' (in a sense that will become clear) in a similar way.

For any $\mathbb{Z}G$-module M, let $M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, the $\mathbb{Z}$-dual of M (with action θ^g: $m \to mg^{-1}\theta$). If we take free presentations

$$B_i \rightarrowtail (\mathbb{Z}G)^{n_i} \twoheadrightarrow A_i^* , \qquad i = 1, 2,$$

and dualise, we obtain

$$A_i \rightarrowtail (\mathbb{Z}G)^{n_i} \twoheadrightarrow B_i^*$$

(since $(\mathbb{Z}G)^* \cong \mathbb{Z}G$), whence

$$A_1 \oplus A_2 \rightarrowtail (\mathbb{Z}G)^n \twoheadrightarrow B_1^* \oplus B_2^*, \quad \text{where } n = n_1 + n_2 .$$

But

$$\bar{R} \rightarrowtail (\mathbb{Z}G)^k \twoheadrightarrow \mathfrak{g} ,$$

by (8), §5 (where k is the rank of the ambient free group to $\bar{R}$) and therefore Schanuel's lemma yields

$$\mathfrak{g} \oplus (\mathbb{Z}G)^n \cong B_1^* \oplus B_2^* \oplus (\mathbb{Z}G)^k .$$

Since A_i is not projective, neither is A_i^* nor B_i^*. Consequently $\mathfrak{g} \oplus (\mathbb{Z}G)^n$ admits a non-projective decomposition.

Definition. Let K be any commutative ring and M a KG-lattice (by which we mean a finitely generated KG-module which is K-projective). We call M a Heller module if any decomposition

$$M \oplus (KG)^n = M_1 \oplus M_2$$

(for any $n \geq 0$) always implies M_1 or M_2 is KG-projective.

Thus we have proved that if $\bar{R}$ is not a Heller module, then neither is $\mathfrak{g}$.

If K is a field (so that the Krull-Schmidt Theorem holds), then M is a Heller module precisely when the uniquely determined largest non-projective direct summand of M is indecomposable. The Heller property is here a generalization of indecomposability. But it is in situations where the Krull-Schmidt Theorem fails that the Heller property really comes into its own as a concept for tackling decomposability problems. We shall see this presently, but for the moment we remark only that if A is a $\mathbb{Z}G$-

lattice and $A = A' \oplus P$ is a decomposition so that P is projective and A' has no projective summand (whence A' is determined by A to within genus), then the Heller property for A implies A' is indecomposable ([8], §8.1; also §12 below).

In all situations, the usefulness of the Heller property depends on the following simple fact (whose proof, with obvious changes in notation, is the argument we had above for $\overline{R}$ and $\mathfrak{g}$):

(11) If $M \rightarrowtail P \twoheadrightarrow N$ is an exact sequence of KG-lattices and P is KG-projective, then M is a Heller module if, and only if, N is a Heller module.

We can now state

(12) There exists a relation module which is not a Heller module if, and only if, $\mathfrak{g}$ is not a Heller module; and this holds if, and only if, $\mathbb{Z}$ is not a Heller module.

Hence, in partial answer to (10), no relation module has a non-projective decomposition if, and only if, $\mathbb{Z}$ is a Heller module. (This uses the fact that a direct sum of a relation module and a free module is another relation module: cf. the beginning of §10 below.)

§8. The prime graph of a finite group

(The material which follows in the rest of this lecture is based on [10] and [8], Lectures 8, 9, as well as on unpublished work of Peter Linnell.)

Suppose $A = M \oplus N$ is a decomposition of an arbitrary $\mathbb{Z}G$-lattice A. If we assume A/pA is a Heller module, then one of the summands, say M/pM, is $\mathbb{F}_p G$-projective. This implies (by standard theory) that $M_{(p)} = M \otimes \mathbb{Z}_{(p)}$ is $\mathbb{Z}_{(p)}G$-projective.

Definition. The $\mathbb{Z}G$-lattice A is a modular Heller module if A/pA is a Heller module for all prime numbers p.

It is obvious that Heller's lemma (11) remains true for modular Heller modules. Since $\mathbb{Z}$ is clearly a modular Heller module, so are all

augmentation ideals and all relation modules.

For any lattice A, let $\pi(A)$ denote the set of all prime numbers p so that $A_{(p)}$ is not projective. Notice that $\pi(A) \leq \pi(G)$ (the set of all prime divisors of the order $|G|$ of G); and that A is projective if, and only if, $\pi(A)$ is empty.

The following result is an elementary consequence of the above definition, notation and remarks:

(13) If A is a modular Heller module and $A = M \oplus N$, then

$\pi(A) = \pi(M) \dot{\cup} \pi(N)$ (disjoint union).

It turns out that $\mathbb{Z}$ is a Heller module for every cyclic group. Suppose that $\mathbb{Z}$ is not a Heller module for G and that $A = \mathbb{Z} \oplus (ZG)^t$ has a non-projective decomposition $A = M \oplus N$. We assume now that G contains a cyclic subgroup C of order pq, where $p \in \pi(M)$, $q \in \pi(N)$. By (13), $p \neq q$. Since $\mathbb{Z}$ is a Heller module for C, one of $M|_C$, $N|_C$ is C-projective. If it is, say $M|_C$, then $M_{(p)}$ is still projective (over $\mathbb{Z}_{(p)}C$) but so also is $N_{(p)}$ because $p \notin \pi(N)$). Then $A_{(p)}$ is projective, which contradicts p in $\pi(A|_C)$. Similarly we obtain a contradiction if $N|_C$ is projective. The conclusion is that G cannot contain any cyclic subgroup like C.

We are now prompted to make the following

Definition. The prime graph $\Pi(G)$ of the group G is the graph with vertices the set $\pi(G)$ and in which p, q are joined by an edge if, and only if, there exists an element in G of order pq. A subset of $\pi(G)$ is called connected if, and only if, it is contained in a connected component of $\Pi(G)$; it is called closed if, and only if, it is the vertex set of a union of connected components.

A small extension of the argument leading to (13) gives the following useful result.

(14) ([8], 8.17) Let A be a modular Heller $\mathbb{Z}G$-module, $\pi(A)$ a closed subset of $\pi(G)$ and assume that for every edge pq in the subgraph (of $\Pi(G)$) determined by $\pi(A)$, there exists a cyclic subgroup C of order

pq so that $A|_C$ is a Heller module and $\pi(A|_C) = \{p, q\}$. If

$$A = A_1 \oplus \ldots \oplus A_k,$$

then each $\pi(A_i)$ is closed and $\pi(A) = \dot{\cup}\, \pi(A_i)$. Moreover, if $\pi(A_i)$ is also connected, then A_i is a Heller module.

Here are some examples of the connectivity of $\Pi(G)$.

(a) If G has a non-trivial centre, or if $G = H \times K$ with H, K non-trivial, then $\Pi(G)$ is connected.

(b) Consider S_n, the symmetric group of degree n and let p be the largest prime $\leq n$. If $p < n - 1$, then $(p + 1, p + 2)$ commutes with $(1, \ldots, q)$ for every q in $\pi(S_n)$, whence 2 is connected to all primes. But if $p \geq n - 1$, then p is isolated. Hence, if $n \geq 3$, $\Pi(S_n)$ is connected if, and only if, n is not of the form p or $p + 1$.

Similarly one sees that if $n \geq 4$, $\Pi(A_n)$ is connected if, and only if, n is not of the form p or $p + 1$ or $p + 2$.

(c) ([8], 8.18) If G is soluble, then $\Pi(G)$ is connected if, and only if, G is not a Frobenius group or a 2-Frobenius group. (We call G a 2-Frobenius group if there is a normal series $1 < H < T < G$ so that T is a Frobenius group with kernel H and G/H is a Frobenius group with kernel T/H.)

In the disconnected case of a soluble group, there are precisely two connected components. It seems that no groups with more than 6 components are presently known.

§9. The main theorem

We come now to the main theorem. This ties together the various concepts we have introduced.

(15) The following conditions on the group G are equivalent:

(i) $\Pi(G)$ is not connected;

(ii) $\mathfrak{g} \oplus \mathbb{Z}G$ has a non-projective decomposition;

(iii) $\mathbb{Z} \oplus (\mathbb{Z}G)^2$ has a non-projective decomposition.

Moreover, $\mathfrak{g}$ decomposes (necessarily non-projectively!) if, and only if, $\mathbb{Z} \oplus \mathbb{Z}G$ has a non-projective decomposition.

The implication (iii) $\Rightarrow$ (i) is immediate from (14); (ii) $\Rightarrow$ (iii) and the last part of the theorem follow from [10], Lemma 3.2. The implication (i) $\Rightarrow$ (ii) is due to Linnell. We make an exceedingly brief comment on the proof.

Assuming (i), let $\pi(G) = \pi \dot{\cup} \pi'$, with π, π' both non-empty and closed. If $\mathfrak{g} \oplus \mathbb{Z}G$ were to admit a non-projective decomposition $A \oplus A'$, with $\pi(A) = \pi$, $\pi(A') = \pi'$, then the characters of A and A' are easy to write down explicitly. Let χ and χ' be two class functions defined precisely like such characters (if they existed). The crux of the proof of (i) $\Rightarrow$ (ii) is to show that χ and χ' actually are characters and that they are the characters of $\mathbb{Z}_\pi$G-projective and $\mathbb{Z}_{\pi'}$G-projective modules respectively. This depends on Swan's 'realization theorems' in [21].

The problem left open by (15) is whether the non-connectivity of $\Pi(G)$ is sufficient to force $\mathfrak{g}$ itself to decompose. This is certainly correct in various classes of groups, among which are all soluble groups and all S_n, A_n. However, in all cases where it is known to be true, a particular decomposition of $\mathfrak{g}$ can be established by using the existence of a special type of subgroup:

Definition. A subgroup H of G is <u>isolated</u> (in G) if H is a proper subgroup, $H \cap g^{-1}Hg = 1$ or H for all $g \in G$ and for all $h \neq 1$ in H, the centralizer of h in G is contained in H.

(16) ([10], Theorem 1) <u>If G contains an isolated subgroup, then $\mathfrak{g}$ decomposes.</u>

Perhaps the existence of an isolated subgroup is equivalent to the decomposability of $\mathfrak{g}$.

§10. Decomposition of relation modules

We return finally to our original question (10) about relation modules.

Suppose $R \rightarrowtail F \overset{\pi}{\twoheadrightarrow} G$ is a free presentation of rank k: i.e. $d(F) = k$. Let us call k the <u>ambient rank</u> of the relation module $\bar{R}$.

If L is free of rank l, then the free presentation (cf. (1), §2)

$$\pi * (\text{collapse}) : F_1 = F * L \twoheadrightarrow G$$

has a relation module isomorphic to $\bar{R} \oplus (\mathbb{Z}G)^l$. By (4), §4, every relation module in F_1 has this form provided only that $l > 0$ when $k = d(G)$. But all relation modules of fixed ambient rank lie in a single genus class (cf. end of §3) and consequently, if the relation module $\bar{R}$ of ambient rank k has a non-projective decomposition, then so has every relation module of ambient rank $\geq k$.

By (15) and (12), we know that there exists a relation module with a non-projective decomposition if, and only if, $\Pi(G)$ is not connected. Hence we have

(17) There exists an integer k so that every relation module of ambient rank $\geq k$ has a non-projective decomposition if, and only if, $\Pi(G)$ is not connected.

The least such integer k can be proved to be at most $2d_G(g)$. It may reasonably be conjectured to be $1 + d_G(g)$: this would imply that if non-projective decompositions occur at all, then they necessarily happen for all non-minimal relation modules. (This is indeed the case for soluble groups.)

Minimal relation modules, however, need not always have non-projective decompositions when the prime graph is non-connected. Suppose G is soluble. We have seen (§8, example (c)) that $\Pi(G)$ is non-connected if, and only if, G is a Frobenius group or a 2-Frobenius group. Contrast this with

(18) ([8], 9.1) If G is a Frobenius group, then the minimal relation modules have non-projective decompositions if, and only if, the Frobenius complements of G are cyclic.

One can also pinpoint the 2-Frobenius groups whose minimal relation modules have a non-projective decomposition. This is not the class of all 2-Frobenius groups as was erroneously asserted in [8], (9.1). Details of the correct result and rectifying the error in [10] will appear (in a paper by Roggenkamp and Gruenberg) in the Proceedings of the London Mathematical Society.

We shall see in the next lecture that for soluble groups, the existence of a non-projective decomposition of a minimal relation module is equiva-

lent to ordinary decomposability.

It is possible to give a criterion for the non-projective decomposability of minimal relation modules valid for all groups of zero generation gap. Let us say that the decomposition $A = B \oplus C$ of the KG-lattice A is <u>additive</u> if $d_{KG}(A) = d_{KG}(B) + d_{KG}(C)$.

(19) <u>If</u> $\mathrm{gap}(G) = 0$, <u>then minimal relation modules of</u> G <u>have non-projective decompositions if, and only if,</u> $\mathfrak{g}_{(G)} \oplus \mathbb{Z}_{(G)}G$ <u>has an additive non-projective decomposition.</u>

(Here $\mathbb{Z}_{(G)} = \bigcap_{p \in \pi(G)} \mathbb{Z}_{(p)}$ and $\mathfrak{g}_{(G)} = \mathfrak{g} \otimes \mathbb{Z}_{(G)}$.)

We shall meet a more satisfactory form of this result in the next lecture (cf. (25)). Unfortunately it is rather difficult to check that a decomposition is additive. It would be very interesting to have a characterization internal to G of the existence of a decomposition as in (19).

III: MINIMAL PROJECTIVE EXTENSIONS

§11. The construction of minimal projectives

Our concern in the last lecture was the possibility of decomposing a projective extension in $(\overset{G}{-})$ so that neither factor is projective. We found that if G is finite, there is no chance of this happening if the prime graph of G is connected. What about factorizations in which we do have a projective factor?

Let $(A\,|\,E)$ be a projective extension and assume

$$(20) \quad (A\,|\,E) \cong (A_1\,|\,E_1)\ \amalg (P\,|\,S),$$

where $(A_1\,|\,E_1)$ is also projective. Then $(P\,|\,S)$ must be a split extension and P is $\mathbb{Z}G$-projective. (Compare this with §6, (9): the proofs are the same: cf. [7], §9.) It may happen that $(A_1\,|\,E_1)$ factorises similarly and we may continue until, perhaps, we reach a projective factor where this process terminates. We call such an extension a <u>minimal projective extension.</u> Thus, the projective extension $(A\,|\,E)$ is minimal if, and only if, any factorization (20) (with $(A_1\,|\,E_1)$ projective) implies $P = 0$.

The group-theoretic procedure for constructing minimal projectives is to start with a free extension $(\overline{R}\,|\,\overline{F})$. (Every projective extension arises as a factor in one such: (9), §6.) We look for a decomposition

$$(21)\quad \overline{R} = A \oplus P,$$

where $\overline{F}$ splits over P and A contains no direct summand over which $\overline{F}$ splits. Then $(A\,|\,\overline{F}/P)$ is a minimal projective extension (and they all arise in this way).

We leave to one side the question whether minimal projectives always exist. They certainly do if a relation module satisfies the ascending chain condition on projective direct summands. In particular, since relation modules embed in free modules (cf. (8), §5), if $\mathbb{Z}G$ is a right noetherian ring, then minimal projectives exist.

When G is finite, they certainly exist. For the rest of this lecture G will again be a finite group. In this situation, projective modules are cohomologically trivial. Hence the above construction of minimal projectives reduces to finding decompositions like (21) subject to P being $\mathbb{Z}G$-projective and A having no projective direct summand.

A natural place to look for minimal projectives is as factors of minimal free extensions (cf. (21) above). But it is not true that every minimal projective extension arises like this. Williams [26] has exhibited a group G having a minimal projective $(A\,|\,E)$ with $d(E) > d(G)$.

§12. Lattice cores and the presentation rank

If L is a ZG-lattice, then a projective excision of L shall be a decomposition $L = L' \oplus P$, where P is projective and L' has no projective direct summand. We call L' an L-core.

There may be many projective excisions, but all L-cores lie in a single genus class; and the projective parts are also all in one genus. Since $\mathbb{Q}P$ is $\mathbb{Q}G$-free (by a theorem of Swan [19]), say

$$\mathbb{Q}P \cong (\mathbb{Q}G)^k,$$

therefore k is an invariant of L, called the projective rank of L and written $pr(L)$. (Cf. [8], Lecture 5 for what follows.)

If $\bar{R}$ is a relation module, we shall call an $\bar{R}$-core a relation core of G. The discussion in the last section has effectively established

(22) The projective extension $(A\,|\,E)$ is minimal if, and only if, A is a relation core of G.

(23) The relation cores of G form a single complete genus class.

In contrast to this, we mention that there are situations when the relation modules of some fixed ambient rank do not constitute a complete genus class. (We know that they all lie within one genus class.)

A natural expectation is that minimal relation modules are relation cores. This is often true but not always.

Definition. If $\bar{R}$ is a minimal relation module, we call $pr(\bar{R})$ (which is an invariant of G) the presentation rank of G and write if $pr(G)$.

Minimal relation modules are relation cores if, and only if, $pr(G) = 0$. Hence, if $\mathbb{Z}$ is a Heller module and $pr(G) = 0$, then every minimal relation module of G is indecomposable.

Let $(\bar{R}\,|\,\bar{F})$ be a minimal free extension. The corresponding relation sequence (cf. (8), §5) is then

$$\bar{R} \rightarrowtail (\mathbb{Z}G)^{d} \twoheadrightarrow \mathfrak{g}, \quad \text{where } d = d(G).$$

If $d' = d_G(\mathfrak{g})$, we have a free module extension

$$A \rightarrowtail (\mathbb{Z}G)^{d'} \twoheadrightarrow \mathfrak{g}.$$

Now

$$A \oplus (\mathbb{Z}G)^{d} \cong \bar{R} \oplus (\mathbb{Z}G)^{d'},$$

whence $A \oplus (\mathbb{Z}G)^{d-d'}$ is in the same genus as $\bar{R}$ (local cancellation being possible). It turns out that d' is also the minimum number of module generators of $\mathfrak{g}_{(G)} = \mathfrak{g} \otimes \mathbb{Z}_{(G)}$, where (as in (19), §10), $\mathbb{Z}_{(G)}$ is the ring of all rational numbers a/b, with b prime to $|G|$. (We return to this point below in §14.) A consequence is that A has no projective

direct summand. Hence $A \oplus (\mathbb{Z}G)^{d-d'}$ is (within genus) a projective excision of $\bar{R}$ and we conclude (using also (23) for (ii))

(24) (i) (Roggenkamp [17]) $pr(G) = gap(G)$;

(ii) A is a relation core of G.

We are now in a position to state a version of (19) (§10) that is valid for all finite groups:

(25) The relation cores of G are decomposable if, and only if, $g_{(G)} \oplus \mathbb{Z}_{(G)}G$ has an additive non-projective decomposition.

§13. The generation gap function

The generation gap (or equivalently, in view of (24), the presentation rank) is a function whose behaviour is rather erratic. We survey some of the known facts about it. (Cf. [8], Lecture 6 and also [1], [9].)

(26) [9] In each of the following situations G has generation gap 0:

(i) $d(G) \leq 2$;

(ii) there exists a soluble normal subgroup N so that $gap(G/N) = 0$

(iii) there exists a normal subgroup N so that G/N is soluble and $d(G) = d(G/N)$.

The proof of (i) is very simple indeed. The result is clear if G is cyclic. Assume that $d(G) = 2$ and that $gap(G) > 0$. Then a projective excision of $\bar{R}$ (a minimal relation module) is $\bar{R} = A \oplus P$, with P projective of rank $k > 0$, whence $\mathbb{Q}\bar{R} = \mathbb{Q}A \oplus (\mathbb{Q}G)^k$. By (8), §5, $\mathbb{Q}\bar{R} \cong \mathbb{Q} \oplus \mathbb{Q}G$. Hence $k = 1$ and $A \cong \mathbb{Z}$ and so (cf. (24)(ii)) $\mathbb{Z}_{(G)} \rightarrowtail \mathbb{Z}_{(G)}G \twoheadrightarrow g_{(G)}$ is exact. Therefore G has projective period 2 and thus is cyclic, contrary to hypothesis.

Note that (ii) implies that all soluble groups have generation gap 0; and that a special case of (iii) is $d(G) = d(G/G')$.

To produce examples of groups with non-zero generation gap, we use direct products. Write $G^{(r)} = G \times \ldots \times G$, with r factors.

(27) (Cossey-Gruenberg-Kovács [1])

$$d_{G^{(r)}}(g^{(r)}) = \max\{d_G(g),\ r\,d(G/G')\}.$$

It follows that if G is perfect (i.e. $G = G'$) then $\mathrm{gap}(G^{(r)})$ increases exactly like $d(G^{(r)})$; and this latter function (at least for simple G) can be studied using methods developed by Hall in 1936 [13]. When G is not perfect ($G > G'$), we apply a theorem of Wiegold [24] that $d(G^{(r)}) = r\,d(G/G')$ for all $r > 2\,d(G)$, to find that $\mathrm{gap}(G^{(r)}) = 0$ for these values of r. Thus we have

(28) $\lim_{r\to\infty} \mathrm{gap}(G^{(r)}) = 0$ or ∞ according as $G \neq G'$ or $G = G'$.

The most useful criterion for deciding whether a group G has non-zero generation gap is representation-theoretic. We know (from (24)) that $\mathrm{gap}(G) > 0$ if, and only if, any one minimal relation module $\bar{R}$ has a non-zero projective direct summand. This is equivalent to the local statement: for each $p \in \pi(G)$, $\mathbb{F}_pG$ is a direct summand of $\bar{R}/p\bar{R}$. Gaschütz [5] has given an explicit description of the modules $\bar{R}/p\bar{R}$ and hence, by using this, we can check whether or not each projective indecomposable $\mathbb{F}_pG$-module occurs in $\bar{R}/p\bar{R}$ at least as often as in the group algebra $\mathbb{F}_pG$. The resulting criterion is the following:

(29) $\mathrm{gap}(G) > 0$ if, and only if, for all $p \in \pi(G)$ and all irreducible $\mathbb{F}_pG$-modules M,

$$|H^1(G, M)| \leq |M|^{d(G)-1-\zeta_M},$$

where

$$\zeta_M = \begin{cases} 1 & \text{if } M \neq \mathbb{F}_p, \\ 0 & \text{if } M = \mathbb{F}_p. \end{cases}$$

It should be noted (and this is an elementary fact) that

$$|H^1(G, M)| \leq |M|^{d(G)-\zeta_M},$$

for all G and all irreducible M.

If we assume that $d(G) > d(S)$ for every composition factor S of G then (29) remains true if M is restricted to those irreducible modules which arise as split chief factors of G ([9], (2.1)). Since all presently known simple groups need at most 2 generators and $gap(G) > 0$ is only possible if $d(G) \geq 3$ (§(26(i)), it follows that the restriction $d(G) > d(S)$ is probably not serious. Nevertheless, even in this form, the criterion is still essentially representation theoretic.

Is there any way of expressing $gap(G) > 0$ in terms only of the internal structure of G? We describe an attempt (as yet unpublished) in this direction.

First some notation and terminology. If S is a composition factor of G, we shall write $A_G(S)$ for the automorphism group induced on S by $N_G(S)$, the normalizer of S in G:

$$A_G(S) = N_G(S)/C_G(S);$$

one calls $A_G(S)$ the automizer of S in G.

Let M be an irreducible module which occurs as a split chief factor of G. If $C = C_G(M)$, the centralizer of M in G, then we can find a chief series of G through C so that all split occurrences of M lie together immediately below C: say $C/D \cong M^{(k)}$ and D contains no split occurrence of M. Let us call k the M-width of G.

Suppose next that G/C is monolithic. This means that G/C has a unique minimal normal subgroup, say L. We assume further that L contains a composition factor whose automizer A (in G) has an irreducible module N so that

$$M \cong N \uparrow_A^G \quad \text{(induced module)}$$

and $End_G(M) \cong End_A(N)$. In these circumstances we shall say that M is monolithically induced.

(30) (Cossey-Gruenberg-Kovács) *Assume that* $d(G) > d(S)$ *for every composition factor* S *of* G. *Then* $gap(G) > 0$ *if, for every split abelian chief factor* M *of* G *of width* $k > 0$, *the following holds:*

if M *is not monolithically induced,* $d(G) - \zeta_M > d_G(M^{(k)})$;

if M is monolithically induced, say $M = N \uparrow_A^G$, where $A = A_G(S)$, then

$$(G : N_G(S))(d(G) - 2) \geq d(N^{(k)} \rfloor A) - 1.$$

The conditions of (30) are also necessary for $\mathrm{gap}(G) > 0$ if the automizer of every non-abelian composition factor of G can be generated by 2 elements.

§14. Swan modules

We have already mentioned that $d_G(g) = d_G(g_{(G)})$ (cf. the argument leading to (24), §12). This is also relevant to the formula for $d_{G^{(r)}}(g^{(r)})$ in (27), §13. What is behind this equality?

Let L be a $\mathbb{Z}G$-lattice. By standard theory (e.g. [8], 7.1) one can prove that

$$(31) \quad d_G(L_{(G)}) = \max\{d_G(L/pL);\ \text{all}\ p \in \pi(G)\}$$

where, of course, $L_{(G)} = L \otimes \mathbb{Z}_{(G)}$; and Swan [22] proved that

$$(32) \quad d_G(L) \leq 1 + d_G(L_{(G)}).$$

We call L a Swan module if, in fact, $d_G(L) = d_G(L_{(G)})$.

Results (31) and (32) illustrate a phenomenon of frequent occurrence. The local information about a lattice can often be tied together in a nice form over the semi-local coefficient ring $\mathbb{Z}_{(G)}$ and this is usually not hard. But the change from $\mathbb{Z}_{(G)}$ to $\mathbb{Z}$ can present formidable difficulties.

Another illustration of this concerns the notion of genus. Two $\mathbb{Z}G$-lattices L, M are in the same genus (i.e. are locally isomorphic) if, and only if, $L_{(G)} \cong M_{(G)}$. This fact is often very useful.

Being a Swan module is not a genus property. For let I be a non-fre projective right ideal in $\mathbb{Z}G$. Then $I_{(G)}$ is free (because all projectives over $\mathbb{Z}_{(G)}G$ are free) and its projective rank must be 1 (cf. §12). Hence $I_{(G)} \cong \mathbb{Z}_{(G)}G$, i.e. I and $\mathbb{Z}G$ are in the same genus. By (32), $d_G(I) \leq 2$. Since $\mathbb{Z}G$ is indecomposable, I cannot be an epimorphic image of $\mathbb{Z}G$,

whence $d_G(I) = 2$.

Swan discovered a useful genus property that ensures a lattice is a Swan module. We define this property (S) as follows:

(33) Definition. The $\mathbb{Z}G$-lattice L has (S) if, and only if $k = d_G(L_{(G)}) \geq 2$ and if $C \rightarrowtail (\mathbb{Z}_{(G)}G)^k \twoheadrightarrow L_{(G)}$ is an exact sequence of $\mathbb{Z}_{(G)}G$-lattices, then $\mathbb{Q}C$ contains a copy of every irreducible $\mathbb{Q}C$-module except possibly $\mathbb{Q}$.

(34) (Swan [22], Lemma 4.4) If the $\mathbb{Z}G$-lattice L has (S), then L is a Swan module.

This result is the basic tool for proving (details in [8], Lecture 7)

(35) All augmentation ideals, all relation modules and all relation cores are Swan modules.

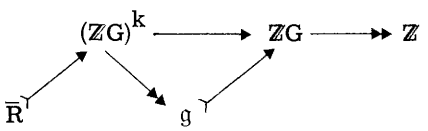
As a consequence we have two noteworthy facts:

(36) ([8], 7.9) (i) If A, B are relation cores, then $d_G(A) = d_G(B)$

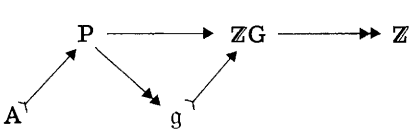
(ii) $d_G(\bar{R}) - d(F)$ is an invariant of G ($\bar{R} \mid \bar{F}$ being any free extension in $\underline{\underline{G}}$)).

§15. Augmented partial Euler characteristics

If $\bar{R}$ is a relation module of ambient rank k then (cf. (8), §5)



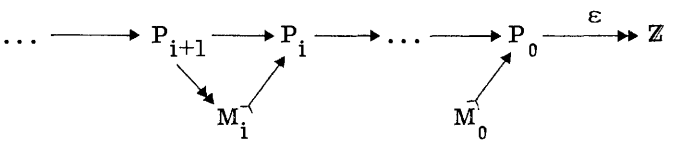
is exact; and if A is a relation core, then we can find a projective $\mathbb{Z}G$-module P (of projective rank $pr(P) = d_G(g)$) so that



is an exact diagram. Thus g, $\bar{R}$, A occur as kernels in projective resolu-

tions of $\mathbb{Z}$. We have seen (cf. (35)) that they are all Swan modules. Is this accidental or do all kernels in all resolutions have this property? This question was answered by Swan in [22]. We survey his results.

Choose a projective resolution of $\mathbb{Z}$:



and abbreviate this as (P, M) or (P). We assume $P_0 = \mathbb{Z}G$ and ε is the unit augmentation (so that $M_0 = \mathfrak{g}$).

Now we define <u>augmented partial Euler characteristics</u> as follows:

$$\nu_n(P) = \mathrm{pr}(P_n) - \mathrm{pr}(P_{n-1}) + \ldots + (-1)^n \mathrm{pr}(P_0);$$

$$\nu_n(G) = \inf\{\nu_n(P);\ \text{all projective resolutions } (P)\};$$

$$\mu_n(G) = \inf\{\nu_n(P);\ \text{all free resolutions } (P)\}.$$

A projective resolution (P) is called <u>minimal</u> if $\nu_n(P) = \nu_n(G)$ for all $n \geq 0$; similarly, a free resolution (P) is called minimal if $\nu_n(P) = \mu_n(G)$ for all $n \geq 0$.

Two resolutions (P), (P') are said to be in the <u>same genus</u> if they become isomorphic when tensored with $\mathbb{Z}_{(G)}$ (cf. §14).

(37) (i) <u>Minimal projective resolutions exist and any two such are in the same genus. Moreover,</u> (P, M) <u>is minimal if, and only if,</u>

$$\mathrm{pr}(P_{i+1}) = d_G((M_i)_{(G)})$$

<u>for all</u> $i \geq 0$.

(ii) <u>Minimal free resolutions exist and any two are in the same genus.</u>

Since we know that $\mathfrak{g}$ is a Swan module, it follows that

$$\nu_1(G) = \mu_1(G) = d_G(\mathfrak{g}) - 1.$$

Also (and here we use (36)(ii)) we have

$\nu_2(G) = \mu_2(G) = d_G(\bar{R}) - d(F) + 1.$

Let (P, M) be a minimal projective resolution. It is then possible to construct a new minimal projective resolution (P', M') so that P'_k is free whenever $M_{k-1} \in (S)$ (cf. (33)). Moreover, if $M_{2m+1} \ncong \mathbb{Z}$, then it can be proved that M_{2m+1} and M_{2m} have the property (S). Note that the condition $M_{2m+1} \cong \mathbb{Z}$ is equivalent to G having periodic cohomology of period dividing $2(m + 1)$.

(38) If G is cyclic or does not have periodic cohomology, then there exists a minimal projective resolution which is free (and therefore minimally free). All the kernels in this resolution are Swan modules.

When G has periodic cohomology of minimal period q, then Swan showed in 1960 [20] that G has a free period lq: this means that there exists a free resolution (E, K) so that $K_i \cong \mathbb{Z}$ for all $i + 1 \equiv 0 \pmod{lq}$. Recently C. T. C. Wall [23] proved that $l \leq 2$; and it seems possible* that $l = 1$ always. In any case, Swan's work [22] yields

(39) Let G have minimum projective period $q > 2$ and minimum free period lq. Suppose (E, K) is any minimal free resolution of $\mathbb{Z}$. Then K_i is a Swan module for all i except when i satisfies

$i + 2 \equiv 0 \pmod{q}$ and $i + 2 \not\equiv 0 \pmod{lq}$.

§16. The presentation deficiency of a direct power

Finally, we return to a problem stated in the first lecture (end of §1): can we ever have $d_G(\bar{R}) < d_F(R)$?

We have seen that $\nu_2(G) = d_G(\bar{R}) - d(F) + 1$. In comparison with (27) we state

(40) If $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, then $\nu_2(G^{(r)}) = \nu_2(G \times G)$ for all $r \geq 2$.

If the homology conditions fail, so that G is not perfect or the Schur multiplier of G is non-zero, or both, then $\lim_{r \to \infty} \nu_2(G^{(r)}) = \infty$. This is straightforward.

* No longer. See problem A3.

Theorem (40) has an interesting consequence for relation cores. Let a_r denote the minimum number of module generators of a relation core of $G^{(r)}$. By (36) (i), this is independent of the particular relation core chosen. Now

$$a_r - d_{G^{(r)}}(\mathfrak{g}^{(r)}) = \nu_2(G^{(r)}) - 1.$$

Assuming G is as in (40), $d_{G^{(r)}}(\mathfrak{g}^{(r)}) = d_G(\mathfrak{g})$, by (27), and therefore

$$a_r = a_2 \quad \text{for all} \quad r \geq 2.$$

There is a possibility that (40) may be used to construct examples of groups G for which $d_G(\bar{R}) < d_F(R)$. We end our account with two suggestions in this direction.

For each $s \geq 1$, let $R_s \rightarrowtail F_s \twoheadrightarrow G^{(s)}$ be a minimal free presentation. If G is a perfect group, then $d(G^{(2^n)}) \leq d(G) + n$ (e.g. [8], 6.18). It follows now, using (40) that

$$\begin{aligned} d_{G^{(2^n)}}(\bar{R}_{2^n}) &= \nu_2(G \times G) - 1 + d(G^{(2^n)}) \\ &\leq \nu_2(G \times G) - 1 + d(G) + n. \end{aligned}$$

Hence the problem is to prove that

$$d_{F_{2^n}}(R_{2^n}) > \nu_2(G \times G) - 1 + d(G) + n = \text{constant} + n.$$

The second approach uses free products. Let $E_r = G * \ldots * G$ (r factors). There is a natural surjection $\tau : E_r \twoheadrightarrow G^{(r)}$ whose kernel E_r^* is a free group. By the Gruško-Neumann theorem [15] we have $d(E_r) = r\,d(G)$. Let $\phi : F \twoheadrightarrow E_r$ be a minimal free presentation of E_r and let $R = \operatorname{Ker} \phi\tau$. Then

$$R \rightarrowtail F \twoheadrightarrow G^{(r)},$$

$R\phi = E_r^*$ and thus

$$d_F(R) \geq d_{E_r}(E_r^*) = e(r), \text{ say.}$$

Assuming G to be as in (40), we obtain

$$d_{G^{(r)}}(\bar{R}) = \nu_2(G \times G) - 1 + r\,d(G),$$

a linear polynomial in r, with non-negative coefficients. The question is whether e(r) increases faster with r than this linear polynomial.

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4 · Arithmetic groups

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This is a survey of results on arithmetic groups. Only a minimal acquaintance with algebraic geometry is assumed. Theorems are mostly quoted without proof: sometimes an indication of the method is given. There is a substantial bibliography, with a guide to the subjects it covers. The reader is referred to the sources therein for the proofs omitted here.

§1. DEFINITIONS AND GENERAL PROPERTIES ([9], [26])

1.1. Let G be an algebraic subgroup of GL_n, defined over the field Q of rational numbers. Thus there exists a set of polynomials (with rational coefficients) in the n^2 matrix entries and the inverse of the determinant, whose set of solutions in any extension E of Q is a subgroup G_E of $GL_n(E)$. We call G_E the group of E-points of G. The groups G_R and G_C are respectively real and complex Lie groups.

We write G_Z for $G_Q \cap GL_n(Z)$.

Definition. A subgroup Γ of G_Q is arithmetic if it is commensurable with G_Z: that is, if $\Gamma \cap G_Z$ has finite index both in Γ and in G_Z.

A group Γ is arithmetic if it can be embedded as an arithmetic subgroup in G_Q for some Q-algebraic subgroup G of GL_n. Then any subgroup of finite index in Γ is also an arithmetic group.

Remarks. (1) We admit all subgroups commensurable with G_Z, rather than G_Z alone, in order to make the definition independent of the chosen Q-embedding of G in a general linear group. Thus a linear algebraic group over Q has a well-defined class of arithmetic subgroups.

(2) One can replace Q in the definition by an arbitrary number field E, and Z by the ring O_E of integers of E; but this does not enlarge the class of 'arithmetic groups'. For let $d = [E : Q]$; then an E-algebraic subgroup H of GL_n determines by 'restriction of scalars' a Q-algebraic subgroup G of GL_{nd}. One can identify G_Q with H_E, and G_Z has finite index in H_{O_E}; so that any subgroup Γ which is arithmetic in H_E (according to the extended definition) is already arithmetic in G_Q according to the definition above.

Functoriality (cf. [41], 10.14, 10.20). Let G and G' be linear algebraic groups over Q, and $\phi : G \to G'$ a homomorphism (defined over Q). Let $\phi_Q : G_Q \to G'_Q$ be the corresponding homomorphism on rational points. Then:

(a) If Γ is arithmetic in G_Q, $\phi_Q(\Gamma)$ is contained in an arithmetic subgroup of G'_Q; it is arithmetic in G'_Q if Coker ϕ is finite.

(b) If Γ' is arithmetic in G'_Q, $\phi_Q^{-1}(\Gamma')$ contains an arithmetic subgroup of G_Q; it is arithmetic in G_Q if Ker ϕ is finite.

(Beware that (a) would not be true for 'congruence subgroups', cf. [43], Sém. Bourbaki.)

1.2 Examples

(1) A finite group is arithmetic.

(2) Let G be the multiplicative group $G_m = GL_1$. Then G_Q is the group Q^* of non-zero rational numbers, and G_Z is $Z^* = \{\pm 1\}$. The arithmetic subgroups of G_m are $\{1\}$ and $\{\pm 1\}$.

(3) Let G be the additive group G_a, so that $G_E = E$ for any E. This group can be embedded in GL_2 as the group of upper unitriangular matrices $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, whose defining equations are $x_{11} = x_{22} = 1$, $x_{21} = 0$. Then $G_Z = Z$; any arithmetic subgroup of G_Q is a subgroup of Q commensurable with Z, so is infinite cyclic.

The class of arithmetic groups is closed under finite products. So by (1) and (3) every finitely generated abelian group is arithmetic.

(4) Every finitely generated torsion-free nilpotent group Γ is arithmetic. In fact, according to Malčev (cf. Bourbaki, LIE II, III, Exercices, pp. 82-3, 281-3) there are embeddings $\Gamma \subset \Gamma_Q \subset \Gamma_R$, where Γ_Q is a uniquely divisible nilpotent group generated by roots of elements

of Γ, and Γ_R is a simply connected nilpotent Lie group in which Γ_Q is dense. Both Γ_Q and Γ_R are unique up to unique isomorphism. With $\Gamma_R \supset \Gamma_Q$ is associated a Lie algebra with a rational structure $\mathfrak{n}_R \supset \mathfrak{n}_Q$. Now Γ_Q can be reconstructed from $\mathfrak{n}_Q$: it is the set $\mathfrak{n}_Q$ with the product given by the Campbell-Hausdorff series

$$x.y = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[x, [x, y]] + \dots$$

which terminates by nilpotency. This multiplication law is polynomial, so Γ_Q becomes a linear algebraic group over $\mathbf{Q}$; and Γ is an arithmetic subgroup.

For example, the nilpotent group with generators a, b and relations $(a, (a, b)) = (b, (a, b)) = 1$ embeds in GL_3 as the group of upper unitriangular matrices over $\mathbf{Z}$: $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$.

(5) Let R be any ring which as a Z-module is free of finite rank. Then the multiplicative group R^* is arithmetic. For let G be the Q-algebraic group whose E-points are those E-automorphisms of the free E-module $R \otimes_{\mathbf{Z}} E$ which are linear with respect to the right R-module structure: then R^*, acting by multiplication on the left, is an arithmetic subgroup of $G_{\mathbf{Q}}$.

In particular, this applies when R is the ring O_K of integers in a number field K, or when R is an order I in a quaternion field over K. Therefore O_K^* and I^* are arithmetic.

(6) Let

$$f(x) = \sum_{i=1}^{n} a_i x_i^2$$

be a non-degenerate quadratic form over $\mathbf{Q}$. Let $SO(n, f)$ be the associated special orthogonal group. Examples of arithmetic subgroups of $SO(n, f)_{\mathbf{Q}}$ are studied in Vinberg [56], [57]. Other semi-simple groups can be used as well. Thus $SL_n(\mathbf{Z})$ and $Sp_{2n}(\mathbf{Z})$ are arithmetic groups associated with Q-split forms of SL_n, Sp_{2n}.

(7) A finitely generated non-abelian free group is arithmetic: indeed such a group is isomorphic to a subgroup of finite index in $SL_2(\mathbf{Z})$. A similar argument, using units of quaternion algebras, shows that the

fundamental group of a compact orientable surface is arithmetic.

(8) Let X be a simply connected finite complex. Then the group Γ of homotopy classes of homotopy equivalences from X to itself is arithmetic. This is due to Sullivan [49] and independently to Wilkerson [61]: it follows from Sullivan's theorem that Γ/C is arithmetic for some finite normal subgroup C ([49](b), 10.3), together with the residual finiteness of Γ, which is a consequence of Sullivan's completion theory ([49](a), 3.2). Further, every arithmetic group occurs (up to commensurability) in this way.

There are similar results concerning $\pi_0(\mathrm{Diff}(M))$, when M is a compact simply connected manifold of dimension at least six ([49](b)).

1.3 Properties of arithmetic groups

In this section, Γ denotes any arithmetic group.

(1) Γ is finitely presented ([8]; [41], 13.15).

(2) Γ has only finitely many conjugacy classes of finite subgroups [8].

(3) Γ is residually finite.

Recall that (3) means that Γ is separated for the topology T of subgroups of finite index. Besides T, it is of interest to consider the congruence topology T_c, in which a basis of neighbourhoods of 1 is given by the kernels of the natural homomorphisms $\Gamma \subset GL_n(Z) \to GL_n(Z/qZ)$, $q \geq 1$. This topology is independent of the chosen embedding $G \subset GL_n$; it is finer than T, and it is obviously separated; hence (3). The congruence subgroup problem asks whether $T_c = T$. In general this is false: $SL_2(Z)$ is a counterexample. But in a number of cases (for instance $SL_n(Z)$, $n \geq 3$, or $Sp_{2n}(Z)$, $n \geq 2$), it is known to be true, or its failure can be measured by a finite group; see [42], [43].

(4) Γ has a torsion-free subgroup of finite index.

This follows from (2) and (3), or, more directly, from Minkowski's theorem [34] that the congruence subgroup of level q of $GL_n(Z)$ is torsion-free if $q \geq 3$.

(5) If Γ is torsion-free, there is a finite complex Y of type $K(\Gamma, 1)$.

That is, there is a pointed finite CW complex (or, equivalently, simplicial complex) Y such that $\pi_1(Y) \approx \Gamma$, and $\pi_i(Y) \approx 0$ for $i \neq 1$ ([14], [40]). Since Γ is finitely presented, (5) is equivalent to the existence of a finite free resolution of the trivial Γ-module Z over the group ring $Z[\Gamma]$:

$$0 \to L_n \to L_{n-1} \to \dots \to L_1 \to L_0 \to Z \to 0$$

where $L_i \approx Z[\Gamma]^{l_i}$, $i \geq 0$. That is, Γ is a group of type (FL) [44]. The Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is defined as $\chi(Y)$, or equivalently as $\sum(-1)^i l_i$.

(6) If Γ is torsion-free, its cohomological dimension $cd(\Gamma)$ is finite, and $H^*(\Gamma; Z)$ is finitely generated [40]. We have $\chi(\Gamma) = \sum(-1)^i (\Gamma; Z)$.

This follows from (5), since $H^*(\Gamma) \approx H^*(Y)$ for any coefficient module.

(7) $H^q(\Gamma; Z[\Gamma])$ is zero except for a single value of q for which it is a free Z-module I [14].

When Γ is torsion-free, the exceptional value is $q = cd(\Gamma)$. In this case Γ is a duality group in the sense of Bieri and Eckmann [7], and its dualizing module is isomorphic to I. The module I has infinite rank in general.

For (8) and (9) below, we assume that the algebraic group G is simple and that $Q\text{-rank}(G) \geq 2$.

(8) Every normal subgroup of Γ either is of finite index, or is finite and central.

(9) Every linear representation of Γ is almost algebraic.

That is, on some subgroup of finite index in Γ it coincides with the restriction of an algebraic representation of G.

This is due to Margulis [30], [31]; see also [1], [35], [39] for special cases.

1.4 The quotient G_R/Γ

If Γ is an arithmetic subgroup of the Q-algebraic group G, then it is a discrete subgroup of the real Lie group G_R. The following

theorem of Borel and Harish-Chandra gives conditions for the homogeneous space G_R/Γ to have finite volume, and for it to be compact. If one of these properties holds for Γ, then it holds for all commensurable subgroups. So the conditions depend only upon the Q-structure of the algebraic group G.

Let G^0 be the connected component of the identity in G, and $X(G^0)$ the group of Q-homomorphisms from G^0 into the multiplicative group G_m.

Theorem ([13]). Let Γ be an arithmetic subgroup of G_Q.

(a) The volume of G_R/Γ is finite if and only if $X(G^0) = \{1\}$.

(b) The following are equivalent:

(i) G_R/Γ is compact

(ii) G has no subgroup isomorphic to G_m

(iii) $X(G^0) = \{1\}$, and every unipotent element of G_Q is contained in the radical.

Remarks. (1) If G is commutative, the conditions for finite volume and for compactness are identical.

(2) If $G = SO(n, f)$, then condition (a) holds unless $n = 2$ and f represents zero (in which case $G \approx G_m$); condition (b) holds if and only if f does not represent zero.

Exercises. (1) Deduce Dirichlet's theorem on units from the compactness criterion above.

(2) Let D be a quaternion field over Q generated by i, j with $i^2 = a$, $j^2 = b$, $ji = -ij$ (for suitable $a, b \in Q$). Show that the arithmetic subgroups of D^* are finite if $a < 0$ and $b < 0$; and otherwise are commensurable with certain discrete subgroups of $SL_2(R)$ having compact quotient.

Generalize to quaternion fields over a number field K.

1.5 Arithmeticity of discrete subgroups of Lie groups

Let L be a real Lie group with a finite number of components, and Γ any discrete subgroup of L. We say Γ is arithmetic in L if there

exist a Q-algebraic group G and a Lie homomorphism $\phi : G_R \to L$ such that

(i) ϕ has compact kernel and open image

(ii) $\phi(\Gamma_1)$ is commensurable with Γ whenever Γ_1 is arithmetic in G_Q.

It suffices to check (ii) for a single choice of Γ_1. If Γ_1 is chosen torsion free, the compactness of the kernel implies that $\phi|\Gamma_1$ is injective.

Theorem (Margulis [30], [31]). *If L is a simple Lie group with $R\text{-rank}(L) \neq 1$, and L/Γ has finite volume, then Γ is arithmetic in L.*

In the case $\mathbf{R}\text{-rank}(L) = 1$ this is not true. Most Fuchsian groups give counterexamples in $SL_2(\mathbf{R})$, and interesting examples in $SO(n, f)$, $n = 4, 5, 6$, have been given by Vinberg [55], [56]. Very recently, an example in $SU(2, 1)$ has been found by Mostow.

§2. ACTION OF Γ ON THE HOMOGENEOUS SPACE $X = K\backslash G_R$

2.1. If G is a Q-algebraic group as above, the real Lie group G_R has finitely many components. It is known that G_R has maximal compact subgroups, and that any two are conjugate. Let K be one of them; the homogeneous space $X = K\backslash G_R$ is diffeomorphic to $\mathbf{R}^d$, where $d = \dim G_R - \dim K$, and G_R acts properly on X. (If G is semisimple, then X is the associated symmetric space.) If Γ is arithmetic in G_Q, it is a discrete subgroup of G_R, so acts properly on X: that is, every compact subset of X meets only finitely many of its translates by elements of Γ (cf. Bourbaki, TG III 32). In particular, the stabilizers of points are finite subgroups. So, if Γ has no torsion, the action is free and $X \to X/\Gamma$ is a Galois covering with group Γ. Since X is a manifold, so is X/Γ. Since X is contractible, X/Γ is a space of type $K(\Gamma, 1)$; that is, $\pi_1(X/\Gamma) \approx \Gamma$ and $\pi_i(X/\Gamma) \approx 0$ for $i \neq 1$.

This implies that, when Γ is torsion-free, the cohomology of the group Γ (with any module of coefficients) is equal to the corresponding cohomology group of the manifold X/Γ. In particular, the cohomological dimension $cd(\Gamma)$ is at most d, where $d = \dim X$ as above. Since X/Γ is a connected d-manifold, we have $cd(\Gamma) = d$ if X/Γ is compact, and

$cd(\Gamma) < d$ if not (see also 2.3).

Remark. The existence of a free action on X yields no more information without a further construction, because Johnson and Wall have shown [58] that every countable group of finite cohomological dimension acts freely on some euclidean space.

If one can find an explicit fundamental domain for the action of Γ on X, then one can find a presentation of Γ, and often also some information about the cohomology of Γ. This is valid whether Γ has torsion or not, and also applies to examples (such as Fuchsian groups) which need not be arithmetic. See for instance [29], [47] for $SL_3(\mathbf{Z})$ and [6], [50] for $SL_2(O_K)$, where K is an imaginary quadratic field.

2.2 Adding corners to X and X/Γ

In the rest of §2, we assume that G is semisimple and connected.

Let Γ be a torsion-free arithmetic subgroup of $G_{\mathbf{Q}}$, and X the homogeneous space $K\backslash G_{\mathbf{R}}$ as in 2.1. If X/Γ is non-compact, it is diffeomorphic to the interior of a compact manifold with boundary. This result, due to Raghunathan [40], implies that Γ is finitely presented and of type (FL) as stated in 1.3. Raghunathan's proof is by construction of a suitable Morse function on X/Γ; it gives no information about the boundary added to X/Γ. A different method was given by Borel and Serre [14]. They show that X is the interior of a manifold with corners $\overline{X}$ which depends upon the Q-structure of G. (A manifold with corners is a Hausdorff space locally modelled upon a product of lines and half-lines $\mathbf{R}^n \times \mathbf{R}_+^m$.) The action of an arithmetic subgroup Γ of $G_{\mathbf{Q}}$ on X extends to a proper action of Γ on $\overline{X}$. If Γ is torsion-free, the quotient $\overline{X}/\Gamma$ is a compact manifold with corners whose interior is X/Γ.

We recall that a subgroup (defined over $\mathbf{Q}$) isomorphic to $G_m \times \ldots \times G_m$ is called a split torus of G. All maximal split tori are conjugate, according to a theorem of Borel and Tits [16]. The dimension $l = l(G)$ of any one of them is called the Q-rank of G.

Exercise. Let $G = SO(n, f)$ as in 1.2(6), with $n \geq 3$. Show that the Q-rank l of G is equal to the dimension of a maximal totally iso-

tropic subspace of f, i.e. that one can write f as $x_1x_2 + x_3x_4 + \ldots + x_{2l-1}x_{2l} + g(x_{2l+1}, \ldots, x_n)$ where g does not represent zero.

The manifold with corners $\overline{X}$ is a disjoint union of subspaces e_P diffeomorphic to $\mathbf{R}^q$, where $d\text{-}l \leq q \leq d = \dim X$. These e_P are indexed by the parabolic subgroups P of G which are defined over Q. (A subgroup P is parabolic if G/P is a projective variety; equivalently, if G_C/P_C is compact.) These subgroups also index the simplices σ_P of the Tits building ([51], [52]) of G. The dimensions of the subspaces e_P, and the incidence relations among their closures, reflect the structure of the building as follows:

$$\dim e_P + \dim \sigma_P = d - 1$$

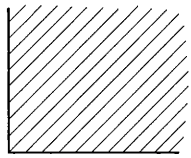
$$e_P \cap \overline{e_Q} \neq \phi \iff e_P \subset \overline{e_Q} \iff \sigma_Q \subset \sigma_P \iff P \subset Q .$$

The minimal parabolic subgroups correspond to the subspaces e_P of dimension $d - l$, and to the maximal simplices of the building. The group G itself corresponds to X, and to the empty simplex. If $l = 0$, then $\overline{X} = X$ and X/Γ is compact (cf. 1.4).

Examples. (1) $G = SL_2$; $l(G) = 1$. The parabolic subgroups are G itself, and all conjugates of the group of upper triangular matrices. The building is discrete and denumerably infinite: $\overline{X}$ is the union of X with a countable number of contractible boundary components (see 2.3).

(2) $G = SL_3$; $l(G) = 2$. The building is a graph, whose vertices are the points and lines of the projective plane over Q. An edge of the graph connects a point and a line when the point lies on the line.

The construction of $\overline{X}$ from X is roughly as follows. For each parabolic subgroup P of G, one defines a 'geodesic action' of the multiplicative group $(R_+^*)^{l(P)}$ on X [14], where $l(P) = \dim \sigma_P + 1$. This action is free, and makes X into a principal $(R_+^*)^{l(P)}$-bundle. The group $(R_+^*)^{l(P)}$ also acts on the product of closed half-lines $(R_+)^{l(P)}$. Let X(P) be the bundle with typical fibre $(R_+)^{l(P)}$ associated with the principal bundle X; it is a manifold with corners. If $P \subset Q$, then $X(Q) \subset X(P)$. We have $X(P) = \bigcup_{Q \supset P} e_Q$. The union of the X(P), as P



runs through all parabolic subgroups, is $\overline{X}$.

When P is a Borel subgroup, the geodesic action of $(R_+^*)^{l(P)}$ can be described in terms of the corresponding decomposition K. A. N. of G_R: one has $X \approx A.N$, and the action is given by multiplication by elements of A.

2.3. Properties of $\overline{X}$ and $\overline{X}/\Gamma$

As above, we assume G is a semisimple connected linear Q-algebraic group. Then $\overline{X}$ has the following properties [14]:

(1) $\overline{X}$ is a Hausdorff manifold with corners, and is countably compact.

(2) The action of G_Q on X extends to an action on $\overline{X}$. (The action of G_R does not in general extend to $\overline{X}$: this reflects the fact that the construction of $\overline{X}$ depends essentially upon the Q-structure of G, and not only upon its R-structure.)

(3) If Γ is arithmetic in G_Q, then Γ acts properly on $\overline{X}$. The quotient $\overline{X}/\Gamma$ is compact, and is a manifold with corners if Γ is torsion-free.

These results subsume various theorems of 'reduction theory', due to Siegel, Borel-Harish-Chandra, Borel etc.; see [9].

(4) The boundary $\partial\overline{X}$ of $\overline{X}$ has the homotopy type of the Tits building T of Q-parabolic subgroups of G. This follows from the fact that the $\overline{e_P}$ $(P \neq G)$ make up a covering of $\partial\overline{X}$ by contractible subsets, whose nerve is isomorphic to T.

(5) $\partial\overline{X}$ has the homotopy type of a bouquet of $(l-1)$-spheres. (For, by a result of Solomon and Tits [46], this is true of the building T. See also [14].) In particular, $\partial\overline{X}$ is connected if the Q-rank l of G is ≥ 2.

Taking reduced homology with Z coefficients, we have by (5)

$$\tilde{H}_i(\partial\overline{X}) \approx \tilde{H}_i(T) \approx \begin{cases} 0 & \text{for } i \neq l-1 \\ I & \text{for } i = l-1 \end{cases}$$

where I is the Steinberg module of G_Q ([14], [46]), which is free as an abelian group. The usual reduced homology groups of $\overline{X}$ are zero, because $\overline{X}$ is contractible. We denote cohomology with compact supports (and integral coefficients) by H_c^*. By Poincaré duality for manifolds with boundary, we have the isomorphism (determined by choice of an orientation for $\overline{X}$)

$$(6) \qquad H_c^q(X) \approx H_{d-q}(\overline{X}, \partial\overline{X}) \approx \begin{cases} 0 & \text{for } q \neq d-l \\ I & \text{for } q = d-l\,. \end{cases}$$

From these results we can deduce properties (5)-(7) of 1.3. Let Γ be a torsion-free arithmetic subgroup of G_Q. Then $\overline{X}/\Gamma$ is a space of type $K(\Gamma, 1)$. It is also a compact manifold with corners, and therefore has the homotopy type of a finite complex (as one sees by smoothing the corners and triangulating, or more easily by Morse theory). Therefore Γ is finitely presented and is a group of type (FL).

Further, the spectral sequence of the covering $\overline{X} \to \overline{X}/\Gamma$, with compact supports and integral coefficients, collapses and yields the isomorphism

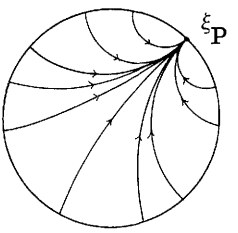
$$H^q(\Gamma; Z[\Gamma]) \approx H_c^q(\overline{X}) \approx \begin{cases} 0 & \text{for } q \neq d-l \\ I & \text{for } q = d-l\,. \end{cases}$$

Therefore Γ is a duality group in the sense of Bieri-Eckmann [7] of dimension $d - l = \dim(K\backslash G_R) - Q\text{-rank}(G)$, and its dualizing module is the Steinberg module I of G_Q.

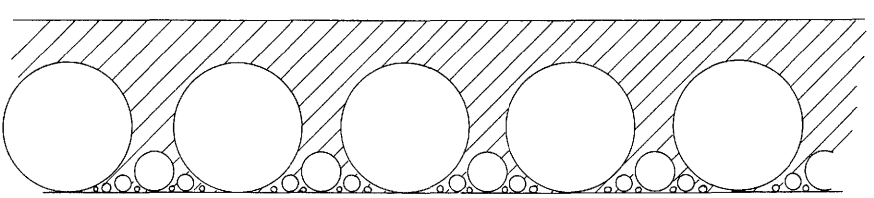
Unfortunately, it is not easy to use the above theory for specific calculations of cohomology. Even the cohomology of $SL_n(Z)$ is not entirely known at present, except for the cases $n = 2$ and $n = 3$ with constant coefficients ([29], [47]).

Example. The case of SL_2. We illustrate the construction of $\overline{X}$ by considering the case $G = SL_2$. Then X is diffeomorphic to the hyperbolic plane, which can be represented as the open unit disc in C, or as the upper half-plane with G_R acting by $z \mapsto \frac{az+b}{cz+d}$, $ad - bc = 1$. Any

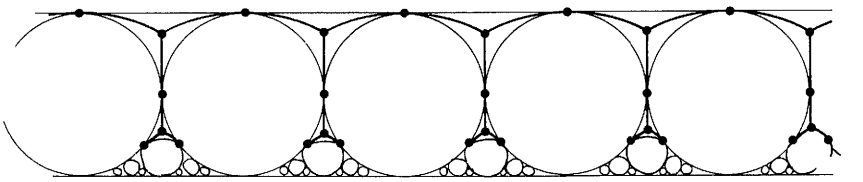
proper parabolic subgroup P over the field Q is the stabilizer of a rational cusp ξ_P on the boundary of the unit disc. In the action of R_+^* on X associated with P, the element $\lambda \in R_+^*$ corresponds to a translation of magnitude $\log \lambda$ along geodesics in the direction of the cusp:



We have $X(P) = X \cup e_P$, where e_P is 'a copy of R at ξ_P'; the points of e_P correspond to the geodesics of X abutting at ξ_P. To visualize this, we take an isometry of the unit disc onto the upper half-plane which throws the cusp ξ_P to infinity. Then R_+^* acts by $(x, y) \mapsto (x, \lambda y)$, so that e_P is a copy of the x-axis added at infinity in the y-direction, and $X(P) = \{(x, y) \mid -\infty < x < \infty,\ 0 < y \leq \infty\}$. One gets a manifold diffeomorphic to $X(P)$ by removing an open collar of its boundary e_P; this gives a representation of $X(P)$ as a strip $\{z \in C \mid 0 < \mathrm{Im}(z) \leq a\}$. Similarly, one can represent the union of all the $X(P)$ as a manifold with boundary contained in the upper half-plane, by removing collar neighbourhoods of all the boundary components. The result looks as follows:



If one widens the excised collars until the boundary components touch, one obtains the closed set of the upper half-plane which is exterior to all the horocycles equivalent under $SL_2(Z)$ to the horocycle $y = 1$:



Although this space is no longer diffeomorphic to $\overline{X}$, the group $SL_2(Z)$ still acts properly on it. The centre $\{\pm 1\}$ acts trivially, and the action preserves the tree shown, whose edges are boundary components of standard fundamental domains. The fundamental domain for the action on the tree is an edge, whose stabilizer in $SL_2(Z)/\{\pm 1\}$ is trivial. The stabilizers of the end-points have orders 2 and 3. From the theory of actions on trees ([45], Ch. 1, §4), we deduce the well-known free product decomposition $SL_2(Z)/\{\pm 1\} \approx (Z/2Z) * (Z/3Z)$.

Remark. The above diagram appears in a different context in Rademacher's work on partitions ([37], p. 267).

§3. EULER-POINCARÉ CHARACTERISTICS ([18], [23], [44])

3.1 The Euler-Poincaré measure

Let G be a semisimple algebraic group over R and $X = K\backslash G_R$ its symmetric space as in §2. Denote the Gauss-Bonnet measure on X by ω_X. There is a unique invariant measure ω_G on G_R whose image under $G_R \to X$ is ω_X; this is the Euler-Poincaré measure of G_R.

Theorem. *Let Γ be a discrete subgroup of G_R. Assume either*

(a) *G_R/Γ is compact;*

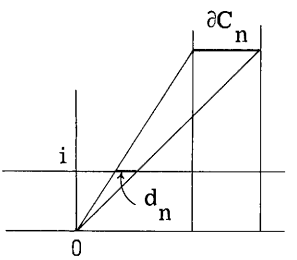
or (b) *Γ is arithmetic in G_R (in the sense of 1.5).*

Then the Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is given by

$$\chi(\Gamma) = \int_{G_R/\Gamma} \omega_G .$$

Here $\chi(\Gamma)$ is taken in the sense of Wall (cf. [18], [44]), i.e. is equal to $\frac{1}{(\Gamma : \Gamma')}\chi(\Gamma')$ where Γ' is a torsion-free subgroup of finite index in Γ. That this is well-defined follows from 1.3. Thus, for the

proof, one may assume Γ to be torsion-free. In case (a), the Gauss-Bonnet theorem gives directly that $\chi(X/\Gamma) = \int_{X/\Gamma}\omega_X = \int_{G_R/\Gamma}\omega_G$, and hence the result since $\chi(\Gamma) = \chi(X/\Gamma)$. Case (b) is a deep result of Harder [23]. The idea of his proof is to take an exhaustion $\{C_n\}$ of X/Γ by compact sets which are manifolds with boundary. From the Gauss-Bonnet formula for C_n one obtains $\chi(C_n) = \int_{C_n}\omega_X + \int_{\partial C_n}\mu$, where μ is some form on ∂C_n. For $n \to \infty$, we have $\lim \int_{C_n}\omega_X = \int_{X/\Gamma}\omega_X$ and $\chi(C_n) = \chi(X/\Gamma)$. Hence one has to prove that $\lim \int_{\partial C_n}\mu = 0$. We



sketch a proof for the case $G = SL_2$ (the same argument works for any group of rank 1). The boundary components of X/Γ correspond to cusps which can be taken to be at ∞. Truncate a cusp neighbourhood as shown and draw a cone from 0 to intersect the line $y = 1$ in d_n. Since μ depends only on the Riemannian structure, which is invariant under dilations, we have $\int_{\partial C_n}\mu = \int_{d_n}\mu$. As $n \to \infty$, ∂C_n travels up the cusp neighbourhood and d_n shrinks to 0. Hence $\lim \int_{d_n}\mu = 0$.

In general, it can be shown that ω_G is non-zero if and only if the Lie groups G_R and K have the same rank. In that case, the sign of the measure is $(-1)^{d/2}$ where $d = \dim X$.

Example. The group SL_n has rank $n - 1$ and its maximal compact subgroup SO_n has rank $[n/2]$. Thus $\omega_G \neq 0$ for $n = 2$, and $\omega_G = 0$ for $n \geq 3$. Let Γ be a torsion-free subgroup of finite index in $SL_3(\mathbb{Z})$. Then by the above $\chi(\Gamma) = 0$. For SL_3, the dimension of the symmetric space is 5 and the Q-rank is 2, so by 2.3 we have $cd(\Gamma) = 3$. Thus $\chi(\Gamma) = 1 - b_1(\Gamma) + b_2(\Gamma) - b_3(\Gamma) = 0$, where b_i is the i^{th} Betti number.

A vanishing theorem of Kajdan (see §4) shows that $b_1(\Gamma) = 0$. The equation then implies $b_3(\Gamma) > 0$.

3.2. Computation of $\chi(\Gamma)$ for Chevalley groups [23]

By Harder's theorem 3.1(b), computing $\chi(\Gamma)$ is equivalent to computing the volume of G_R/Γ with respect to ω_G. This has been done for Chevalley groups. More precisely:

Theorem (Harder). Let G be a simply-connected simple Chevalley group over $\mathbf{Z}$. Let $\Gamma = G(\mathbf{Z})$.

Then $\chi(\Gamma) = c \prod_{i=1}^{l} \zeta(1 - m_i)$ where l is the rank of G, $m_1, \ldots, m_l$ are the degrees of the fundamental invariants of W_G, $c = |W_G|/2^l |W_K|$, W_G, W_K being the Weyl groups of G, K respectively and $\zeta(s)$ the Riemann zeta function.

(Recall that, if m is an integer ≥ 2, $\zeta(1 - m)$ is a rational number, viz. $-b_m/m$ where b_m is the m^{th} Bernoulli number; it is non-zero if and only if m is even.)

Harder's proof uses Langlands' theorem that the Tamagawa number of G is equal to 1; see [23] for the details.

Examples. 1. When $G = SL_2$, we have $|W_G| = 2$, $|W_K| = 1$, $c = 1$, $m_1 = 2$. Hence $\chi(SL_2(\mathbf{Z})) = \zeta(-1) = -b_2/2 = -1/12$ (as was a priori obvious from the decomposition of $SL_2(\mathbf{Z})/\{\pm 1\}$ as a free product).

2. When $G = Sp_{2n}$, we have $l = n$, $c = 1$, and $m_i = 2i$ for $1 \leq i \leq n$. Hence

$$\chi(Sp_{2n}(\mathbf{Z})) = \prod_{i=1}^{n} \zeta(1 - 2i) = \prod_{i=1}^{n} (-b_{2i}/2i).$$

Remark. More generally, if E is a totally real number field of degree r, with ring of integers O_E, then

$$\chi(G(O_E)) = c^r \prod_{i=1}^{l} \zeta_E(1 - m_i)$$

where ζ_E is the zeta-function of the field E. (Note that, if E is not totally real, then $\chi(\Gamma) = 0$.)

Exercise. Prove the following equivalences:

$$\chi(G(Z)) \neq 0 \iff -1 \in W_G \iff \text{all the } m_i\text{'s are even.}$$

3.3. Finite p-subgroups of Chevalley groups

Let G be a simple simply connected Chevalley group (cf. 3.2). We want to compute, as far as possible, the maximal order of a finite p-group contained in $G(Q)$, or $G(Z)$.

(a) An upper bound for the order of a p-subgroup of $G(Q)$.

We follow a method of Minkowski [34]:

Let F be a p-subgroup of $G(Q)$. Choose $N \geq 1$ such that F is contained in $G(Z[\frac{1}{N}])$. If q is a prime not dividing N, F can be reduced mod q, and the reduction homomorphism $F \to G(F_q)$ is easily shown to be faithful if $q \neq p$. Hence, $|F|$ divides the order of $G(F_q)$, which is:

$$|G(F_q)| = q^n \prod_{i=1}^{l} (q^{m_i} - 1),$$

where l, m_i are as in Harder's theorem (3.2), and $n = \frac{1}{2}(\dim G - l)$.

If $p \neq 2$, we choose q such that its class $(\mathrm{mod}\ p^2)$ generates the multiplicative group $(Z/p^2Z)^*$; we then have

$$v_p(q^m - 1) = \begin{cases} 1 + v_p(m) & \text{if } p - 1 \text{ divides } m \\ 0 & \text{if not,} \end{cases}$$

where v_p is the p-adic valuation.

If $p = 2$, we choose q such that $q \equiv 3 \pmod 8$; we have

$$v_2(q^m - 1) = \begin{cases} 2 + v_2(m) & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

If we denote by $m(G, p)$ the p-adic valuation of $|G(F_q)|$, we see that

$$|F| \leq p^{m(G, p)}$$

and

$$m(G,\ p) = \sum_{m_i \equiv 0 \ (\mathrm{mod}(p-1))} (1 + v_p(m_i)) \qquad (p \geq 3)$$

$$m(G,\ 2) = l + \sum_{m_i \text{ even}} (1 + v_2(m_i)) \qquad (p = 2).$$

By Sylow's theorem, the order of any finite subgroup of $G(Q)$ divides $M_G = \prod_p p^{m(G,\ p)}$.

Examples.

G_2 - $(m_i) = 2,\ 6$ - $M_G = 2^6 . 3^3 . 7$

F_4 - $(m_i) = 2,\ 6,\ 8,\ 12$ - $M_G = 2^{15} . 3^6 . 5^2 . 7^2 . 13$

E_6 - $(m_i) = 2,\ 5,\ 6,\ 8,\ 9,\ 12$ - $M_G = 2^{17} . 3^{10} . 5^2 . 7^2 . 13$

E_7 - $(m_i) = 2,\ 6,\ 8,\ 10,\ 12,\ 14,\ 18$ - $M_G = 2^{24} . 3^{11} . 5^2 . 7^3 . 11 . 13 . 19$

E_8 - $(m_i) = 2, 8, 12, 14, 18, 20, 24, 30$ - $M_G = 2^{30} . 3^{13} . 5^5 . 7^4 . 11^2 . 13^2 . 19 . 31$.

(b) A lower bound for the order of a p-subgroup of $G(Z)$

Let Γ be a group of type VFL. If $\chi(\Gamma) \neq 0$ and p^N appears in the denominator of $\chi(\Gamma)$, then by a theorem of K. Brown ([18], see also K. Brown's lectures), Γ contains a subgroup of order p^N.

For $\Gamma = G(Z)$, $\chi(\Gamma)$ is computable by Harder's theorem. It is non-zero for G_2, F_4, E_7, E_8 but is zero for E_6. Thus, one obtains

$G_2(Z)$, $F_4(Z)$, $E_7(Z)$, $E_8(Z)$ has subgroups of orders

$2^6,\ 3^2,\ 7$

$2^{13},\ 3^5,\ 5^2,\ 7^2,\ 13$

$2^{21},\ 3^9,\ 5^2,\ 7^3,\ 11,\ 13,\ 19$

$2^{30}, 3^{10}, 5^4, 7^4, 11^2, 13^2, 19, 31$

(c) Comparison of (a) and (b)

We limit ourselves to the case of E_8 (for E_7, see K. Brown's lectures). By comparing (a) and (b), we see that the maximal order of a p-subgroup of $E_8(Q)$ is:

1 when $p \neq 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 19,\ 31$;

p when $p = 19$ or 31;

p^2 when $p = 11$ or 13;

p^4 when $p = 7$;

p^4 or p^5 when $p = 5$;

p^{10}, p^{11}, p^{12} or p^{13} when $p = 3$;
p^{30} when $p = 2$.

Exercises. 1. Let F and F' be two p-subgroups of $G(\mathbb{Q})$ of order $p^{m(G,p)}$. Show that F and F' are isomorphic (use reduction (mod q) as in (a) above, and apply Sylow's theorem to $G(\mathbb{F}_q)$). Show that one can choose an isomorphism $\phi : F \to F'$ such that x and $\phi(x)$ are conjugate in $G(\bar{\mathbb{Q}})$ for every $x \in F$.

This shows, for instance, that the subgroups of $E_8(\mathbb{Q})$ of order 2^{30} are isomorphic to each other.

2. Let C be a subgroup of $E_8(\mathbb{Q})$ of order 31. Prove:

(i) There is a unique Cartan subgroup T of E_8 containing C; it is the centralizer of C.

(ii) The torus T splits over the field K of 31^{st} roots of unity. The corresponding homomorphism $\mathrm{Gal}(K/\mathbb{Q}) \to W_{E_8}$ is faithful; its image is generated by a Coxeter element of W_{E_8}.

(iii) The group C is contained in a Frobenius subgroup of order $30 \cdot 31$ of $E_8(K)$.

(Problem: Is it true that C acts on each fundamental module of E_8 by a multiple of the regular representation? Note that the dimension of such a module is divisible by 31, cf. Bourbaki, LIE VIII, §9, exerc. 4.)

3. Same questions as in exercise 2, with $(E_8, 31)$ replaced by $(E_7, 19)$ and $(F_4, 13)$.

§4. VANISHING THEOREMS AND LINEAR REPRESENTATIONS ([12], [17], [28], [32], [41])

4.1. Kajdan's Theorem ([28])

Let G be a locally compact separable group, and let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of G (of any dimension: finite or infinite). There is a natural topology on $\hat{G}$, cf. [17], [28]. For instance, if $G = \mathbb{R}$, $\hat{G}$ is homeomorphic to $\mathbb{R}$; if G is compact, $\hat{G}$ is discrete.

Theorem (Kajdan). Assume G has the following property:

(T) - The unit representation 1 is isolated in $\hat{G}$.

If Γ is a discrete subgroup of G such that G/Γ has finite volume, then Γ is finitely generated, and $b_1(\Gamma) = 0$ (hence $\Gamma/(\Gamma, \Gamma)$ is finite).

This is proved by showing that Γ inherits property (T), and that, for a discrete group, (T) implies finite generation and $b_1 = 0$, cf. [28].

Corollary. If G is a real simple Lie group of rank ≥ 2, and if Γ is an arithmetic subgroup of G, then $b_1(\Gamma) = 0$.

Indeed, one can show that such a group G has property (T), cf. [28].

Remark. Kajdan's theorem can also be applied to discrete subgroups of p-adic Lie groups, or more generally products of such groups by real Lie groups (compare §5).

4.2. Connections between cohomology and linear representations ([12], [17])

Let G be a semi-simple real Lie group, K a maximal compact subgroup of G, and $X = K\backslash G$ the corresponding symmetric space (cf. §2). Denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$ respectively.

Let Γ be a discrete torsion-free subgroup of G. Assume that G/Γ is compact; the same is then true for X/Γ. We have

$$H^*(\Gamma; C) \simeq H^*(X/\Gamma; C),$$

which, by de Rham's theory, is isomorphic to the cohomology of the complex of differential forms on X/Γ. This complex is, in turn, isomorphic to the cochain complex $C^*(\mathfrak{g}, \mathfrak{k}\ ; C^\infty(G/\Gamma))$ giving the relative Lie algebra cohomology of $(\mathfrak{g}, \mathfrak{k})$ with values in the space $C^\infty(G/\Gamma)$ of smooth functions on G/Γ. We thus have

$$H^*(\Gamma; C) \simeq H^*(\mathfrak{g}, \mathfrak{k}\ ; C^\infty(G/\Gamma)).$$

By a theorem of van Est (cf. [12]), this is isomorphic to $H_c^*(G; C^\infty(G/\Gamma))$, where $H_c^*(G; E)$ denotes the Eilenberg-MacLane cohomology of G with values in the G-module E, using smooth cochains (or continuous ones, this amounts to the same).

Since G/Γ is compact, $L^2(G/\Gamma)$ is a hilbertian direct sum of closed G-irreducible subspaces, with finite multiplicities:

$$L^2(G/\Gamma) \simeq \hat{\oplus}_{\Pi \in \hat{G}} m(\Pi, \Gamma) M_\Pi .$$

Now, $C^\infty(G/\Gamma)$ is contained in $L^2(G/\Gamma)$, and contains the algebraic direct sum $\oplus\, m(\Pi, \Gamma) M_\Pi^\infty$, where M_Π^∞ is the space of smooth vectors of M_Π. This suggests that

$$H_c^*(G, C^\infty(G/\Gamma)) \simeq \oplus_\Pi\, m(\Pi, \Gamma) H_c^*(G, M_\Pi^\infty),$$

and this is what one can indeed prove ([12], [17]); moreover, only a finite number of terms in the above direct sum are non-zero.

Putting all this together, we get

$$H^*(\Gamma; C) \simeq \oplus_\Pi\, m(\Pi, \Gamma) H_c^*(G; M_\Pi^\infty).$$

Notice that the groups $H_c^q(G; M_\Pi^\infty)$ depend only on G and Π, not on Γ. Many results on these groups have been proved recently by Borel, Casselman, Wallach, Zuckerman, ... (cf. [17]). For instance:

(1) $H_c^*(G; M_\Pi^\infty) = 0$ if the infinitesimal character of M_Π is non trivial.

(2) $H_c^q(G; M_\Pi^\infty) = 0$ for $q < \mathrm{rank}_{\mathbf{R}}\, G$ if $\mathrm{Ker}(\Pi)$ is compact.

Assume G is simple and non compact. Then only the trivial representation 1 has non compact kernel; since it occurs with multiplicity 1, we obtain

$$H^q(\Gamma; C) \simeq \oplus_\Pi\, m(\Pi, \Gamma) H_c^q(G; M_\Pi^\infty)$$
$$\simeq H_c^q(G; C) \simeq H^q(\mathfrak{g}, \mathfrak{k}; C) \text{ for } q < \mathrm{rank}_{\mathbf{R}}\, G.$$

Now $H^*(\mathfrak{g}, \mathfrak{k}; C)$ is well-known to be isomorphic to the cohomology of the compact dual X^* of X. (If $G = SL_2(\mathbf{R})$, then X is the Poincaré half-plane, and X^* the complex projective line.) Thus:

Theorem. If G is simple and non compact, and if Γ is a discrete subgroup of G with compact quotient, then

$$H^q(\Gamma; C) \simeq H^q(X^*; C) \text{ for } q < \operatorname{rank}_{\mathbf{R}} G.$$

This is called a vanishing theorem, since it implies, for instance, that $b^q(\Gamma) = 0$ for any odd $q < \operatorname{rank}_{\mathbf{R}} G$. One may reformulate it as follows (see [32], [38]):

Let H^q denote the space of harmonic q-forms on X which are G-invariant. It is easy to see that $H^q \simeq H^q(X^*; C)$ for all q. On the other hand, every element of H^q defines a harmonic q-form on X/Γ, hence an element of $H^q(X/\Gamma; C) = H^q(\Gamma; C)$. We thus obtain a map $H^q \to H^q(\Gamma; C)$, which is easily seen to be injective. Thus the above theorem is equivalent to saying that this map is surjective in the range $q < \operatorname{rank}_{\mathbf{R}} G$.

Remark. When G/Γ is non compact, and Γ is arithmetic, the above isomorphism $H^q(\Gamma; C) \simeq H^q(X^*; C)$ holds for $q < c(G)$ where $c(G)$ depends only on G, and is approximately $\frac{1}{4}\operatorname{rank}_{\mathbf{R}} G$ ([11], [17]). Since $c(G) \to \infty$ with $\operatorname{rank}_{\mathbf{R}} G$, one can then compute the stable cohomology of the arithmetic subgroups of SL, SO, Sp, and this in turn has applications to the K-theory of rings of integers of number fields (Borel [11]).

§5. S-ARITHMETIC GROUPS ([10], [15], [44])

5.1. Definition and main properties (number field case)

Let E be a number field, and S a finite set of non-zero prime ideals of the ring O_E of integers of E. An element x of E is called an S-integer if $v_p(x) \geq 0$ for all $p \notin S$, where v_p denotes the discrete valuation of E defined by the prime p. Let $O_{E,S}$ be the ring of S-integers of E, and let G be an algebraic subgroup of GL_n defined over E; put $G(O_{E,S}) = G_E \cap GL_n(O_{E,S})$.

Definition. A subgroup of G_E is S-arithmetic if it is commensurable with $G(O_{E,S})$.

As in §1, this definition is independent of the chosen embedding of G in a linear group.

Examples. 1. When $S = \emptyset$, we have $O_{E,S} = O_E$ and 'S-arithmetic' means 'arithmetic'.

2. Take $E = Q$, $S = \{p_1, \ldots, p_k\}$. Then $O_{E,S} = Z[1/m]$, where $m = p_1 \ldots p_k$. Notice that an S-arithmetic subgroup of the additive group G_a is commensurable with $Z[1/m]$, hence is not finitely generated if $k \geq 1$; this explains why, in the theorem below, we assume that the algebraic group G is semi-simple.

3. In case S is the set of all primes of O_E dividing $p_1, \ldots, p_k$ one may use restriction of scalars (as in 1.1, Remark (ii)) to replace E by Q, and S by $\{p_1, \ldots, p_k\}$. However, not all sets S are of this simple form; this is why we cannot keep to the case $E = Q$, as we did in the arithmetic case.

Theorem (Borel-Serre [15]). Let G be a semi-simple algebraic group over E, and let Γ be an S-arithmetic subgroup of G_E. Then Γ has properties (1) to (7) of 1.3.

Corollary. If Γ is torsion-free, it is a finitely presented duality group of type (FL).

The proofs of these results use in an essential way the Bruhat-Tits buildings of G at the primes p of S (see 5.2 and 5.3 below).

Remark. The above theorem holds, more generally, when G is a reductive group, i.e. is isogenous to the product of a semi-simple group and a group of multiplicative type.

5.2. The Bruhat-Tits building ([19], [20])

This is a local construction: we start from a local field E_p with finite residue field (hence locally compact), and a simply connected

(semi)simple algebraic group G over E_p. Bruhat-Tits associate with these data a building X_p which has many properties in common with the symmetric space $K\backslash G_R$ of the real case (§2):

(a) X_p is a contractible (poly)simplicial complex whose dimension is the E_p-rank of G. In particular, $H^q(X_p; Z) = 0$ for $q \neq 0$.

(b) The locally compact group G_{E_p} acts properly on X_p, with fundamental domain a (poly)simplex. The stabilizers of the vertices of X_p are the maximal compact subgroups of G_{E_p}.

(c) The cohomology with compact supports of X_p is given by:

$$H_c^q(X_p; Z) = \begin{cases} 0 & \text{if } q \neq \operatorname{rank}_{E_p} G \\ I_p & \text{if } q = \operatorname{rank}_{E_p} G, \end{cases}$$

where I_p is a Z-free module on which G_{E_p} acts by the 'Steinberg representation' (see [12], [15]).

Example: SL_2. When $G = SL_2$, X_p is of dimension 1; by (a), it is a contractible 1-complex, i.e. a tree. Let us sketch its construction, assuming for simplicity that E_p is the p-adic field Q_p (for more details, see [45]):

Let V denote the vector space Q_p^2, on which $SL_2(Q_p)$ acts in a natural way. The vertices of X_p are equivalence classes of lattices (i.e. Z_p-submodules of V of rank 2), two lattices L and L' being equivalent if there exists $\lambda \in Q_p^*$ such that $\lambda L = L'$. Two vertices are joined by an edge if they have representatives L, L' with $L \supset L'$ and $(L : L') = p$. Every vertex belongs to $p + 1$ edges. For $p = 2$, the tree X_p looks like the diagram on page 128.

One may take for fundamental domain an edge $\overset{L}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\overset{L'}{\circ}$. The stabiliser of the first vertex L is $SL_2(Z_p)$, embedded in the usual way in $SL_2(Q_p)$; the stabiliser of L' is:

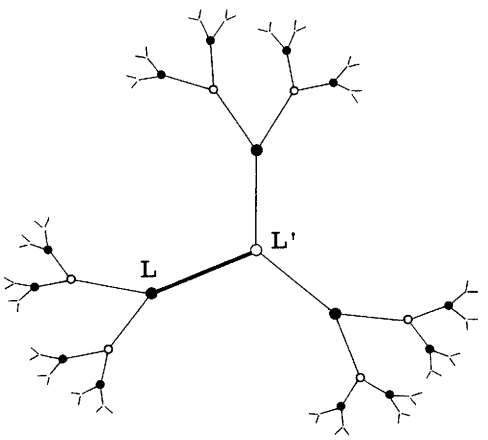
$$SL_2(Z_p)' = \left\{ \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix} \;\middle|\; a, b, c, d \in Z_p,\ ad - bc = 1 \right\}.$$

The stabiliser of the edge LL' is $SL_2(Z_p) \cap SL_2(Z_p)' = \hat{\Gamma}_0(p)$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_p,\ ad - bc = 1 \text{ and } c \equiv 0 \pmod{p} \right\}$$

which has index $p + 1$ in both $SL_2(\mathbb{Z}_p)$ and $SL_2(\mathbb{Z}_p)'$. It follows that

$$SL_2(\mathbb{Q}_p) \simeq SL_2(\mathbb{Z}_p) *_{\hat{\Gamma}_0(p)} SL_2(\mathbb{Z}_p)', \quad \text{cf. [27], [45].}$$



Applications. (i) Let Γ be a discrete torsion-free subgroup of $SL_2(\mathbb{Q}_p)$. Then Γ acts freely on X_p since the stabiliser of any vertex is both discrete and compact, hence finite. It then follows that Γ is isomorphic to the fundamental group of the graph X_p/Γ; hence Γ is a free group (Ihara's theorem, cf. [27], [45]).

(ii) The group $SL_2(\mathbb{Z}[1/p])$ is dense in $SL_2(\mathbb{Q}_p)$, so that the edge LL' is a fundamental domain for its action on X_p. The stabiliser of L is $SL_2(\mathbb{Z})$, and the stabiliser of L' is a conjugate $SL_2(\mathbb{Z})'$ of $SL_2(\mathbb{Z})$. Thus

$$SL_2(\mathbb{Z}[1/p]) = SL_2(\mathbb{Z}) *_{\Gamma_0(p)} SL_2(\mathbb{Z})',$$

where $\Gamma_0(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{p} \}$.

This implies, for instance, that

$$\text{vcd } SL_2(\mathbb{Z}[1/p]) = 2 \text{ and } \chi(SL_2(\mathbb{Z}[1/p])) = (p - 1)/12.$$

Exercise. Let Γ be a torsion-free subgroup of finite index of $SL_2(\mathbb{Z}[1/p])$.

(a) Show that $\Gamma \simeq F_1 *_F F_2$, where the F_i's are free non abelian groups, and F has finite index in both (use the fact that Γ is dense in $SL_2(Q_p)$).

(b) By th. 3 of [43], p. 500, every non trivial normal subgroup of Γ is of finite index. Deduce that Γ is not SQ-universal. (Recall that a group H is SQ-universal if every countable group is isomorphic to a subgroup of a quotient of H.)

Thus an amalgam of free non abelian groups need not be SQ-universal

Problem. Is it true that $SL_2(Z[1/p])$ is coherent, i. e. that each of its finitely generated subgroups is finitely presented? One may ask the same question for $SL_3(Z)$.

5. 3. Applications of the Bruhat-Tits buildings to cohomology ([15], [44])

As in 5. 2, let Γ be an S-arithmetic subgroup of G_E, where G is semi-simple. It is easy to see that Γ is a discrete subgroup of the locally compact group

$$G_S = \prod_{v \text{ arch.}} G_{E_v} \times \prod_{p \in S} G_{E_p},$$

where E_v is the completion of E at the archimedean place v (hence E_v is isomorphic to R or to C). If we denote by H the group $R_{E/Q}G$ deduced from G by restriction of scalars from E to Q (cf. 1.1), we may rewrite G_S as:

$$G_S = H_R \times \prod_{p \in S} G_{E_p} .$$

The group G_S, hence also Γ, acts in a natural way on the space

$$X_S = \overline{X}_R \times \prod_{p \in S} X_p ,$$

where X_p ($p \in S$) is a Bruhat-Tits building as above, and $\overline{X}_R$ is the manifold with corners associated with H (cf. §2).

Theorem ([15]). The group Γ acts properly on X_S, and the quotient X_S/Γ is compact.

Thus, if Γ is torsion-free, it acts freely on X_S, and we have

$$\pi_1(X_S/\Gamma) \simeq \Gamma, \quad \pi_i(X_S/\Gamma) = 0 \text{ for } i \neq 1, \quad H^*(X_S/\Gamma) \simeq H^*(\Gamma).$$

Using the compactness of X_S/Γ, this shows that Γ is finitely presented of type (FL). The fact that Γ is a duality group comes from the vanishing of the cohomology with compact supports of X_S in all dimensions except one, where it is Z-free; this dimension is:

$$\begin{aligned} cd(\Gamma) &= \dim X_S - \operatorname{rank}_Q H = \dim X_S - \operatorname{rank}_E G \\ &= \dim \overline{X}_R - \operatorname{rank}_E G + \sum_{p \in S} \operatorname{rank}_{E_p} G . \end{aligned}$$

Euler-Poincaré characteristics. On each G_{E_p}, there is a unique invariant measure μ_p such that

$$\sum_\sigma (-1)^{\dim \sigma} / \mu_p(G_{p,\sigma}) = 1,$$

where the summation is over a set of representatives of the cells σ of X_p modulo G_{E_p}, and $G_{p,\sigma}$ is the stabiliser of σ. This measure is called the Euler-Poincaré measure of G_{E_p}; it is > 0 (resp. < 0) if $\operatorname{rank}_{E_p} G$ is even (resp. odd), cf. [44]. If we put on $G_S = H_R \times \Pi\, G_{E_p}$ the product of the Euler-Poincaré measure of H_R (cf. §3) and of the μ_p ($p \in S$), we get an invariant measure μ_S.

Theorem ([44]). $\chi(\Gamma) = \mu_S(G_S/\Gamma)$.

This is proved by induction on $|S|$, starting from the case $|S| = 0$, which is Harder's theorem 3.1.

Exercises. 1. Show that $\mu_p = \sum (-1)^{\dim \sigma} \mu_{p,\sigma}$, where $\mu_{p,\sigma}$ is the unique Haar measure on G_{E_p} for which $G_{p,\sigma}$ has volume 1.

2. Show that the cohomological dimension of a torsion-free subgroup of $SL_n(Z[1/p_1 \ldots p_k])$, where the p_i's are primes, is $\leq (n-1)(k + \frac{n}{2})$. Use this to prove that a finitely generated torsion-free

subgroup of $GL_n(\mathbf{Q})$ has finite cohomological dimension ([44]).

5.4. The function field case ([3], [5], [15], [44], [45], [48])

Let k be a finite field, C a complete non-singular curve over k with function field E, and S a finite non-empty set of closed points of C. Let $O_{E,S}$ be the subring of E consisting of functions having no poles outside S. Let G be an algebraic group over E. As in 5.1, one says that a subgroup of G_E is S-arithmetic if it is commensurable with $G(O_{E,S})$.

Example. If C is the projective line P_1, and $S = \{\infty\}$, then $E = k(t)$, $O_{E,S} = k[t]$; if we take $G = SL_n$, we thus see that an S-arithmetic subgroup of $SL_n(E)$ is a subgroup which is commensurable with $SL_n(k[t])$.

For $n = 2$, it is known (see below) that $SL_n(k[t])$ is not finitely generated. This shows that, even when G is semi-simple, S-arithmetic subgroups of G_E can be rather pathological. There is one case, however, where they behave quite well:

Theorem ([15], [44]). Assume that G is semi-simple, and $\mathrm{rank}_E G = 0$. Then every S-arithmetic subgroup Γ of G_E has properties (1) to (7) of 1.3.

To prove this, one first observes that Γ acts properly on the product X_S of the buildings X_p ($p \in S$), and that the quotient X_S/Γ is compact (this is where the hypothesis on $\mathrm{rank}_E G$ is used). Properties (1), (2), (3), (4) follow easily from this, and (5), (6), (7) are proved as in the number field case.

When $\mathrm{rank}_E G \geq 1$, one has only scattered results. For instance:

(i) $SL_3(k[t])$ is finitely generated, but not finitely presented ([5]).

(ii) $SL_2(O_{E,S})$ is finitely generated if and only if $|S| \geq 2$, and it is finitely presented if and only if $|S| \geq 3$ ([36], [43], [45], [48]).

For the cohomology of such groups, see Harder [25].

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5 · Topological methods in group theory

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Introduction

This article is a revised version of notes on an advanced course given in Liverpool from January to March 1977 in preparation for the symposium. The lectures given by Terry Wall at the symposium were mainly taken from Sections 3 and 4, and much of the material in John Stallings' lectures is in Sections 5 and 6. It seemed worth publishing the whole, as a rather full introduction to the area for those with a background in topology. Originality is not claimed for the results in the earlier sections (though full references have not always been given), but the uniqueness results in Section 7 and most of Section 8 are due to Peter Scott.

1. BASIC NOTIONS

The link between topology and group theory comes from the fundamental group. I shall make no attempt to present this: almost every introductory topology text does so. Particularly suitable for this course is Massey's book [18]. An equivalent account, from a different viewpoint, is given by Brown [2]. Let us recall the basic properties of the fundamental group.

(1) For every topological space X and point $x \in X$ we have a group $\pi_1(X; x)$. This depends only on the path component of X containing x. A path from x to y induces an isomorphism $\pi_1(X; x) \to \pi_1(X; y)$; a closed path induces an inner automorphism. A map $f : X \to Y$ with $f(x) = y$ induces a homomorphism $f_* : \pi_1(X; x) \to \pi_1(Y; y)$, and this assignment is functorial: in fact we have a homotopy functor.

(2) A map $\pi : Z \to X$ is a <u>covering</u> if it is locally trivial, with discrete fibres - i.e. every $x \in X$ has a neighbourhood U, and a homeo-

morphism $\pi^{-1}(U) \xrightarrow{h} U \times D$ with D discrete, such that $pr_1 \circ h = \pi$. For a covering π, and $z \in Z$, the map $\pi_* : \pi_1(Z; z) \to \pi_1(X; f(z))$ is injective. If X is reasonably nice (the minimal technical conditions are path-connected, locally path-connected, and weakly locally 1-connected), the correspondence between triples ($\pi : Z \to X$ a connected covering, $z \in \pi^{-1}(x)$) and subgroups $\pi_*\pi_1(Z; z)$ of $\pi_1(X; x)$ induces an isomorphism of categories. Hence, in particular, the coverings are isomorphic if and only if the corresponding subgroups are conjugate.

(3) The third basic fact we need to recall is the technique for calculating fundamental groups, due to van Kampen. First suppose X_1, X_2 are path connected open subsets of X, with path-connected intersection X_0. Then for any base point $x \in X_0$, we have the following commutative diagram in which all the maps are induced by inclusions of spaces.

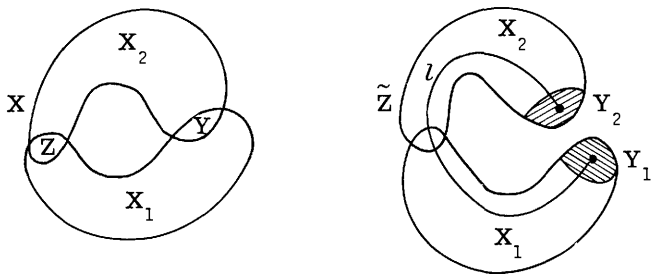
$$\begin{array}{ccc} \pi_1(X_0; x) & \xrightarrow{i_1} & \pi_1(X_1; x) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \pi_1(X_2; x) & \xrightarrow{j_2} & \pi_1(X; x) \end{array}$$

Proposition 1.1. This is a pushout diagram in the category of groups. In other words, for any group G we have a bijection, induced by (j_1, j_2), $\mathrm{Hom}(\pi_1(X; x), G) \cong \{(f_1, f_2) \in \mathrm{Hom}(\pi_1(X_1; x), G) \times \mathrm{Hom}(\pi_1(X_2; x), G): f_1 i_1 = f_2 i_2 \in \mathrm{Hom}(\pi_1(X_0; x), G)\}$.

The standard argument with universals shows that the pushout is uniquely determined; existence is provided by the proposition. The proof, involving breaking up a path in X into subpaths each lying in X_1 or X_2, is somewhat messy. The restriction that the X_i are open can be relaxed if each is a deformation retract of some neighbourhood, as is usually the case in practice.

The restriction that X_0 be connected is less desirable. One may reformulate the proposition to cover this case by using groupoids [2]. More naively, suppose X_0 has just two path-components Y and Z; define $\tilde{Z}$ by identifying X_1 and X_2 along Z, and $\tilde{X}$ by attaching $Y \times I$ to $\tilde{Z}$ by identifying $Y \times i$ to the copy Y_{i+1} of Y in X_{i+1} $(i = 0, 1)$.

There is an obvious map $\phi : \tilde{X} \to X$ (identify $Y \times I$ to Y) which is usually a homotopy equivalence, and induces isomorphisms of π_1. We can calculate $\pi_1(\tilde{Z})$ by Proposition 1.1.



Now choose a base point $y \in Y$; let y_i be the corresponding point in Y_i, and l a path in $\tilde{Z}$ joining y_1 to y_2. We have homomorphisms

$$\alpha_1 : \pi_1(Y; y) \cong \pi_1(Y_1; y_1) \to \pi_1(\tilde{Z}; y_1),$$
$$\alpha_2 : \pi_1(Y; y) \cong \pi_1(Y_2; y_2) \to \pi_1(\tilde{Z}; y_2) \overset{l}{\cong} \pi_1(\tilde{Z}; y_1).$$

Proposition 1.2. For any group G, we have a bijection

$$\mathrm{Hom}(\pi_1(\tilde{X}; y_1), G) \cong \{(f_1, t) \in \mathrm{Hom}(\pi_1(\tilde{Z}; y_1), G) \times G :$$
$$f_1\alpha_2(p) = t^{-1} f_1 \alpha_1(p) t \text{ for all } p \in \pi_1(Y; y)\}.$$

Here, f_1 is the composite $\pi_1(\tilde{Z}; y_1) \to \pi_1(\tilde{X}; y_1) \to G$, and t is the image of the class in $\pi_1(\tilde{X}; y_1)$ of the loop $l \cup y \times I$.

We will show that Proposition 1.2 follows from Proposition 1.1.

In order to start our study of fundamental groups, we need to know that the circle S^1 has infinite cyclic fundamental group. This is easy to prove using the covering of S^1 by the real line $\mathbb{R}$. One might think of deducing this result from Proposition 1.2 by taking X to be S^1 and X_1 and X_2 to be open intervals. For then $\mathrm{Hom}(\pi_1(X), G) \cong G$ for any group G. However our proof of Proposition 1.2 uses the fact that S^1 has infinite cyclic fundamental group. Here is a quick sketch of that proof.

First, the intermediate space $W = \tilde{Z} \cup (y \times I)$ can be considered as the union of $\tilde{Z}$ and the circle $l \cup (y \times I)$ intersecting in the arc l.

Hence $\pi_1(\tilde{W}, y_1)$ is the pushout of $\mathbb{Z} \leftarrow 1 \rightarrow \pi_1(\tilde{Z}, y_1)$, i.e. (see later) the free product $\mathbb{Z} * \pi_1(\tilde{Z}, y_1)$. Now $\tilde{X} = W \cup (Y \times I)$, and $W \cap (Y \times I) = (Y \times 0) \cup (y \times I) \cup (Y \times 1)$. We deduce, after a little manipulation, a pushout diagram

$$\begin{array}{ccc} \pi_1(Y, y) * \pi_1(Y, y) & \xrightarrow{\phi} & \mathbb{Z} * \pi_1(\tilde{Z}, y_1) \\ \downarrow & & \downarrow \\ \pi_1(Y, y) & \longrightarrow & \pi_1(\tilde{X}, y_1) \end{array}$$

where ϕ is given by α_1 on the first factor and by $c \mapsto t \cdot \alpha_2(c) \cdot t^{-1}$ on the second. This is equivalent to Proposition 1.2. We have given the proposition independently, however, since it introduces a construction which will be important below.

Example 1.3. $X = S^1 \vee S^1$, the one-point union. Now apply Proposition 1.1, taking X_1, X_2 as the two circles. Thus $\mathrm{Hom}(\pi_1(X;y), G) \cong G \times G$. The group $\pi_1(X, y)$ is called the free group on generators t_1, t_2 the classes of the circles.

To put some bones into this abstraction, we next give a concrete description of this free group on t, u. A letter is any one of t, u, $\bar{t}$, $\bar{u}$. A word is a finite (perhaps empty) sequence of letters. The word is reduced if none of $t\bar{t}$, $\bar{t}t$, $u\bar{u}$, $\bar{u}u$ occurs as a pair of consecutive letters.

Theorem 1.4. *There is a bijection between elements of the free group F on t, u and the set W of reduced words. Each word defines an element of G by forming the product of various of t, u, t^{-1}, u^{-1} in the indicated order.*

Proof. (i) Observe that F contains the elements t, u and hence the set H of products of finite ordered sequences of elements t, u, t^{-1}, u^{-1}. Clearly H is closed under products and inverses, hence is a subgroup. There is a homomorphism $F \rightarrow H$ such that $t \rightarrow t$, $u \rightarrow u$. The composite $F \rightarrow H \subset F$ coincides with the identity on t, u hence is the identity. Thus $F = H$.

By definition, each element of H is represented by a word. If the word is not reduced, we can cancel products tt^{-1} etc. Thus each element is represented by a reduced word and we have a surjection $\alpha : W \to F$. It remains to prove α bijective.

(ii) Write S for the symmetric group on the set W of reduced words. Define permutations $\tau, \vartheta \in S$ as follows: If the word w ends in $\bar{t}$ (resp. $\bar{u}$), then $w\tau$ (resp. $w\vartheta$) is obtained from w by deleting the last letter. Otherwise, $w\tau$ (resp. $w\vartheta$) consists of w followed by t (resp. u). We see at once that these are permutations with inverses τ^{-1}, ϑ^{-1} defined similarly but interchanging the roles of t and $\bar{t}$ (resp. u and $\bar{u}$).

By definition of F, there is a unique homomorphism $\phi : F \to S$ such that $\phi(t) = \tau$, $\phi(u) = \vartheta$. We define a map $\beta : F \to W$ by $\beta(g) = \phi(g)(1)$. For any reduced word w, we see by induction on the length of w that $\beta(\alpha(w)) = w$. Thus α is injective, hence bijective.

We used the example of $S^1 \vee S^1$ to demonstrate the existence of a free group F of rank two. Note that the proof above does this quite independently for it shows that the set W has a natural group structure which makes it a free group of rank two.

There is an obvious analogue to the above for the free group $F(X)$ on any set X of generators. If X is finite, existence is seen by induction. We observe that if $X_1 \subset X_2$, the natural map $F(X_1) \to F(X_2)$ is injective. Now for X infinite, define

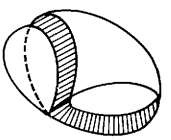
$$F(X) = \cup \{F(Y) : Y \subset X,\ Y \text{ finite}\}.$$

If $y_i \in F(Y_i) \subset F(X)$ for $i = 1, 2, 3$ we define $y_1 y_2$ to be the product in $F(Y_1 \cup Y_2)$. Associativity follows by considering $F(Y_1 \cup Y_2 \cup Y_3)$. It is immediately verified that for any G, restriction to X yields a bijection $\mathrm{Hom}(F(X), G) = \mathrm{Map}(X, G)$, so $F(X)$ is th free group on X.

Now consider any group G, set X and map $\phi : X \to G$. By the above, ϕ has a unique extension $\psi : F(X) \to G$ to a homomorphism whose image is then a subgroup $\overline{X}$. Any element of $\overline{X}$ can be written as a word in the elements $\phi(x)$, and thus lies in any subgroup of G containing $\phi(X)$. Thus $\overline{X}$ is the intersection of the subgroups containing $\phi(X)$: it is called

the subgroup generated by $\phi(X)$ (or by ϕ). If $\overline{X} = G$, we say G is generated by $\phi(X)$.

Now consider a finite CW-complex K with one vertex x. By induction we see that the 1-skeleton K^1 has free fundamental group generated by the classes g_i of the 1-cells. As K^1 is a deformation retract of a neighbourhood, we can apply van Kampen's theorem to calculate the effect on the fundamental group of attaching a 2-cell e^2. Now e^2 is contractible, and a suitable neighbourhood of K^1 meets it in a copy of $S^1 \times \mathbb{R}$. The map $\alpha : S^1 \to N(K^1) \to K^1$ is homotopic to the attaching map of the cell, and determines $\alpha_* : \mathbb{Z} = \pi_1 S^1 \to \pi_1 K^1$ with $\alpha_*(1) = r$, say.



Then $\pi_1(K^1 \cup e^2)$ is the pushout of

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\alpha_*} & \pi_1(K^1) \\ \downarrow & & \\ \{1\} & & \end{array}$$

Arguing similarly with the other 2-cells e_j, we find that if their attaching maps yield classes $r_j \in \pi_1(K^1)$, then the 2-skeleton K^2 has fundamental group $\pi_1(K^2)$ characterized by the property that for any group G,

$$\begin{aligned} \mathrm{Hom}(\pi_1(K^2), G) = \{ & f \in \mathrm{Hom}(\pi_1(K^1), G): \\ & f(r_j) = 1 \text{ for each } j \}. \end{aligned}$$

Since $\pi_1(K^1)$ is free on $\{g_i : i \in I\}$, f is determined by the $f(g_i)$.

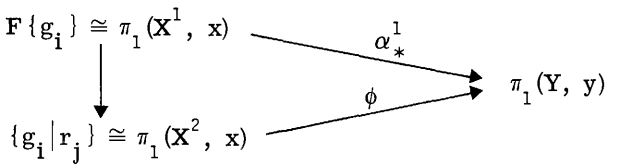
The sequence $\{g_i | r_j\}$ where the g_i are abstract symbols and $r_j \in F\{g_i\}$ is called a presentation of π if, for any G, $\mathrm{Hom}(\pi, G)$ is given as above. The same argument as in (i) of the proof above shows that the images of the g_i are generators of π. The r_j are called

relators. Let N be the subgroup of $F\{g_i\}$ generated by the r_j and all their conjugates: N is called the normal closure of the r_j. Clearly it is the least normal subgroup of $F\{g_i\}$ containing them all. Hence $F\{g_i\}/N$ has the universal property defining $\{g_i|r_j\}$: this yields a construction of this group. Again, the restriction to finite sets of generators and relators is easily seen to be irrelevant.

Of course you have all seen generators and relators before: here it is the relation with two dimensional CW complexes that I wish to stress. (Incidentally, adding cells of dimension > 2 does not affect π_1, as we see on applying van Kampen's theorem again.) For example, let X^2 be any such - say having one vertex x - and Y any space.

Lemma 1.5. *For any homomorphism* $\phi : \pi_1(X^2, x) \to \pi_1(Y, y)$ *there is a map* $\alpha : X^2 \to Y$ *with* $\alpha_* = \phi$.

Proof. The 1-cells and 2-cells of X^2 give a presentation $\pi_1(X^2) = \{g_i|r_j\}$. The image of g_i by ϕ is an element of $\pi_1(Y)$, represented by a map $(S^1, x) \to (Y, y)$. Use these maps to define $\alpha^1 : X^1 \to Y$. Then we have a diagram



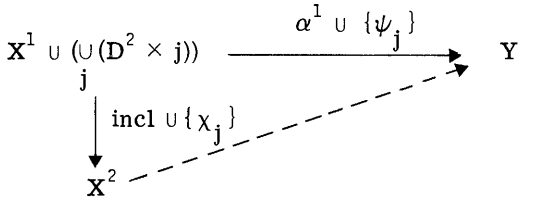
which commutes, by construction of α^1. Hence $\alpha^1_*(r_j) = 1$. For each 2-cell of X, with characteristic map

$$\chi_j : (D^2, S^1) \to (X^2, X^1, x)$$

the class of $\chi_j|S^1$ is r_j (by definition) so the class of $\alpha^1 \circ \chi_j|S^1$ is $\alpha^1_*(r_j) = 1$. Thus $\alpha^1 \circ \chi_j$ is nullhomotopic, so there is a continuous extension

$$\psi_j : D^2 \to Y$$

with $\psi_j|S^1 = \alpha^1 \circ (\chi_j|S^1)$. Now by definition of the topology of X as an identification space, the diagram



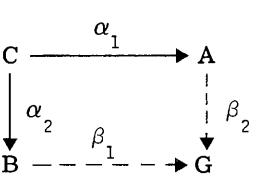
defines a map $\alpha : X^2 \to Y$ such that $\alpha | X^1 = \alpha^1$ and $\alpha \circ \chi_j = \psi_j$. Since $\pi_1(X^2, x)$ is a quotient of $\pi_1(X^1, x)$ it follows that $\alpha_* = \phi$.

Remarks. The condition that X has dimension 2 is essential here. However, a particularly interesting case is when Y is such that $\pi_1(Y, y) = \pi$ and for any (X, x) the map $\alpha \to \alpha_*$ gives a bijection between homotopy classes of maps $(X, x) \to (Y, y)$ and $\mathrm{Hom}(\pi_1(X, x), \pi)$. Such a (Y, y) is called a classifying space for π (or Eilenberg-MacLane space $K(\pi, 1)$). The usual argument with universals shows its uniqueness (up to homotopy); existence is also not hard to prove, for any π. However, the existence of a Y which is e. g. a manifold, or finite complex imposes an interesting and subtle condition on π.

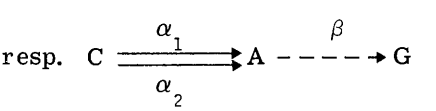
I now return to Propositions 1. 1 and 1. 2. We have obtained in some cases fairly explicit descriptions of the groups so defined. I will now give some useful generalizations of these.

The real beginning of our subject was the discovery that in the case when the maps are injective one can obtain structure theorems similar to 1. 4. In fact a description of $A *_C B$ in terms of reduced words was already given by Schreier in 1927 [36], but a more thorough account and the start of the recent work of the subject is contained in Hanna Neumann's thesis [35]. The groups $A *_C$ go back to [13], though they are not actually defined in that paper. There have of course been many papers on the subject since; our account derives from those of Serre [22] and Cohen [5] and seems simpler and more natural than the original papers.

Propositions 1. 1, resp. 1. 2 concerned diagrams



resp.



Definition. If the maps α_1, α_2 are injective, the universal group G is called the free product of A and B amalgamated along C (resp. amalgamated free product of A along C), and denoted by $A *_C B$ resp. $A *_C$. If C is the trivial group, then $A *_C B$ is denoted by $A * B$ and is called the free product of A and B.

Note. There seems no reason except tradition for calling the first an amalgamated product but the second an HNN group.

We now present the traditional combinatorial arguments for analysing the structure of amalgamated products. Essentially equivalent results will be obtained below independently by geometrical reasoning.

In each of the above cases, one can give an explicit description using reduced words. The easy first half of the proof of Theorem 1.4 shows (with only slight changes) that any element of $A *_C B$ can be written as a product

$$a_1 b_1 a_2 b_2 \cdots a_n b_n \text{ with } a_i \in \beta_1(A),\ b_i \in \beta_2(B) \text{ (maybe } = 1)$$

and that any element of $A *_C$ can be written as

$$a_1 t^{r_1} a_2 t^{r_2} \cdots a_n t^{r_n} \text{ with } a_i \in \beta(A),\ r_i \in \mathbb{Z}.$$

Now we restrict our attention to $A *_C B$. Again, it is clear that some reduction is possible - e.g. for $c \in C$, $\beta_1(a)\beta_2(\alpha_2(c)\cdot b) = \beta_1(a\cdot \alpha_1(c))\beta_2(b)$. We deal with this by pushing all the c's to the right, as follows. To simplify notation, write α_1, α_2 as inclusions so $C \subset A$, $C \subset B$. Pick representatives $a_i \in A$ for the right cosets $a_i C$ of C - thus giving a section of the projection $A \to A/C$, or right transversal of C in A; then do the same for B. We impose the restriction that the identity coset C is represented by the identity element.

A reduced word is now a sequence

$$a_1 b_1 \cdots a_n b_n c$$

such that $c \in C$, a_i belongs to the chosen transversal T_A for C in A, b_i belongs to the chosen transversal T_B for C in B and

$a_i = 1 \Rightarrow i = 1$, $b_i = 1 \Rightarrow i = n$.

Any element of $A *_C B$ may be represented by a reduced word. For write the element as

$$a_1 b_1 \ldots a_n b_n$$

and use induction on n. For $n = 0$, the empty word may be represented by $1 \in C$: a reduced word. Otherwise, by inductive hypothesis, we may write

$$a_1 b_1 \ldots a_{n-1} b_{n-1} = a'_1 b'_1 \ldots a'_r b'_r c', \text{ a reduced word with } r \leq n-1.$$

If now $b'_r = 1$ resp. $a_n \in C$ then $(a'_r c' a_n) \in A$ resp. $(b'_r c' a_n b_n) \in B$ and we have a word of length $\leq r$, which may be reduced by inductive hypothesis. Otherwise write $c' a_n = a'_{r+1} c''$ with $1 \neq a'_{r+1} \in T_A$ and $c'' b_n = b'_{r+1} c$ with $b_{r+1} \in T_B$ and we have a reduced word $a'_1 b'_1 a'_2 \ldots b'_{r+1} c$.

Theorem 1.6. The maps $A \to A *_C B$, $B \to A *_C B$ are injective: every element may be represented by a unique reduced word.

Proof. Again write W for the set of reduced words.

Define an action of B on W by

$$(a_1 b_1 \ldots a_n b_n c)^b = a_1 \ldots a_n b' c' \text{ if } b_n c b = b' c' \text{ in } B \text{ with } b' \in T_B.$$

To check that this is an action, observe that the part of the word up to (and including) a_n is left fixed; for the words $a_1 \ldots a_n b_n c$ which start so, it is equivalent to the right action of B on itself.

Define an action of A on W by

$$(a_1 b_1 \ldots a_n b_n c)^a = \begin{cases} a_1 \ldots b_n a' c' & \text{if } b_n \neq 1,\ ca = a'c' \text{ with } a' \in T_A \\ a_1 \ldots b_{n-1} a'' c'' & \text{if } b_n = 1,\ a_n ca = a''c'' \text{ with } a'' \in T_A \end{cases}$$

To check that this is an action, observe that the part of the word up to b_n (if $b_n \neq 1$) or b_{n-1} (if $b_n = 1$) is fixed; the rest is the standard right action. This defines maps $A \to S(W)$, $B \to S(W)$ which clearly agree on

C, hence define $A *_C B \xrightarrow{\phi} S(W)$. It is now immediate by induction again that $(\ )\phi(w) = w$ for any reduced word w, so these elements of the group are distinct.

For $A *_C$ we proceed similarly. Pick right transversals T_i of $\alpha_i(C)$ in A. Now $t\alpha_2(c) = \alpha_1(c)t$ so $t^{-1}\alpha_1(c^{-1}) = \alpha_2(c^{-1})t^{-1}$ and we define a reduced word to be one of the form

$$a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \dots a_n t^{\varepsilon_n} a_{n+1}$$

where $\varepsilon_i = \pm 1$, $a_i \in T_1$ if $\varepsilon_i = +1$, $a_i \in T_2$ if $\varepsilon_i = -1$ and moreover $a_i \neq 1$ if $\varepsilon_{i-1} \neq \varepsilon_i$. We let a_{n+1} be arbitrary. The above relations allow us to bring any word to a reduced form.

Theorem 1.7. The map $\beta : A \to A *_C$ is injective. Every element is represented by a unique reduced word.

Proof. Again we define an action. The element $a \in A$ acts by sending the final a_{n+1} to $a_{n+1}a$; t corresponds to the permutation τ defined by setting

$$(a_1 t^{\varepsilon_1} \dots a_n t^{\varepsilon_n} a_{n+1})\tau =$$

$$a_1 t^{\varepsilon_1} \dots a_{n-1} t^{\varepsilon_{n-1}} (a_n \alpha_2 \alpha_1^{-1}(a_{n+1})) \text{ if } \varepsilon_n = -1 \text{ and } a_{n+1} \in \alpha_1(C)$$

$$a_1 t^{\varepsilon_1} \dots a_n t^{\varepsilon_n} a'_{n+1} t\alpha_2(c') \text{ otherwise, where}$$

$$a_{n+1} = a'_{n+1}\alpha_1(c') \text{ with } a'_{n+1} \in T_1 .$$

We see that this is a permutation by verifying that an inverse is given by

$$(a_1 t^{\varepsilon_1} \dots a_n t^{\varepsilon_n} a_{n+1})\bar{\tau} =$$

$$a_1 t^{\varepsilon_1} \dots a_{n-1} t^{\varepsilon_{n-1}} (a_n \alpha_1 \alpha_2^{-1}(a_{n+1})) \text{ if } \varepsilon_n = +1 \text{ and } a_{n+1} \in \alpha_2(C),$$

$$a_1 t^{\varepsilon_1} \dots a_n t^{\varepsilon_n} a''_{n+1} t^{-1} \alpha_1(c'') \text{ otherwise, where}$$

$$a_{n+1} = a''_{n+1}\alpha_2(c'') \text{ with } a''_{n+1} \in T_2 .$$

The proof now concludes as before.

2. GRUSKO'S THEOREM

Grušhko's Theorem. Let F be a finitely generated free group, $G = G_1 * G_2$ and let $\phi : F \to G$ be an epimorphism. Then there are subgroups F_1 and F_2 of F such that $F = F_1 * F_2$ and $\phi(F_i) = G_i$.

This is a subtle result about generators of G. It says that if G can be generated by n elements, then there exists a set of n generators for G with each element in G_1 or G_2. This gives us the inequality

$$\mu(G) \geq \mu(G_1) + \mu(G_2)$$

where $\mu(G)$ denotes the minimal number of generators of a group. But the reverse inequality is obvious, so we deduce

Corollary 2.1. If $G = G_1 * G_2$, then $\mu(G) = \mu(G_1) + \mu(G_2)$.

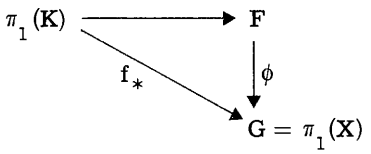
As only the trivial group can have μ equal to zero, we see that $\mu(G_i) < \mu(G)$ when G_1 and G_2 are nontrivial.

Corollary 2.2. If G is a finitely generated group, then $G = G_1 * \ldots * G_n$ for some n, where each G_i is indecomposable. (I.e. $G_i = A * B$ implies A or B is trivial.)

We now give Stallings' proof [25] of Grušhko's Theorem. See [12] for a proof using groupoids and see [3] for a proof using Bass-Serre theory (Chapter 4 of these notes).

Pick two CW-complexes with fundamental groups G_1 and G_2 and construct a CW-complex X with fundamental group $G_1 * G_2$ by joining these two complexes with an interval E. Let v denote the midpoint of E and subdivide E so that v is a vertex of E. We will take v as the basepoint of X. Let X_i denote the closure of the component of $X - \{v\}$ whose fundamental group is G_i.

Let K be a based space and let $f : K \to X$ be a based map. We will say that f represents ϕ if there is an isomorphism of $\pi_1(K)$ with F such that the diagram below commutes.



We consider 2-dimensional CW-complexes K and cellular maps $f : K \to X$ which represent ϕ. Such maps certainly exist for one can take K to be the wedge of n circles where n is the rank of F. However, this particular choice of K may not be the correct one for our purposes. Our aim is to choose K and a map $f : K \to X$ representing ϕ so that $f^{-1}(v)$ is a tree. Once this is achieved, the result follows easily. For let L_i denote $f^{-1}(X_i)$ and let F_i denote $\pi_1(L_i)$. As $L_1 \cup L_2 = K$, and as $L_1 \cap L_2 = f^{-1}(v)$ which is simply connected, we see that $\pi_1(K) = F_1 * F_2$. Also $f_*(F_i) \subset G_i$ as $f(K_i) \subset X_i$. Now the results of Theorem 1.6 on reduced words in a free product, show that we must have $f_*(F_i) = G_i$, because ϕ is an epimorphism. This is now the conclusion of Grušhko's Theorem.

We find an appropriate choice for K by starting with a space K_0 and a map $f_0 : K_0 \to X$ representing ϕ and then performing a sequence of modifications to K_0 and f_0. We do this so as to obtain spaces $K_1, K_2, \ldots$ and maps $f_1, f_2, \ldots$ all representing ϕ, such that $f_n^{-1}(v)$ is a forest (disjoint union of trees) with α_n components and $\alpha_{n+1} < \alpha_n$, for $n \geq 0$. After at most α_0 steps, we will obtain a space K_n and map $f_n : K_n \to X$ representing ϕ such that $f_n^{-1}(v)$ is a tree, and the result will follow.

We take K_0 to be the wedge of n circles, where n is the rank of F and choose a cellular map $f_0 : K_0 \to X$ representing ϕ so that $f_0^{-1}(v)$ is a finite number of 0-cells in K_0. In particular, $f_0^{-1}(v)$ is a forest with α_0 components. If $f_0^{-1}(v)$ is connected (and hence a single point), we already have the space K and map f which we want. Otherwise, we use the following lemma to construct the sequence of spaces already described and hence deduce the result.

Lemma 2.3. *Let K be a based CW-complex and $f : K \to X$ a map representing $\phi : F \to G$ such that $f^{-1}(v)$ is a forest with α components. If $\alpha \geq 2$, then there is a based CW-complex K' and a map $f' : K' \to X$*

representing ϕ such that $f'^{-1}(v)$ is a forest with $\alpha - 1$ components.

Proof. Let l be a path in K with endpoints in $f^{-1}(v)$. (By a path, we mean simply that l is a map $I \to K$.) Then $f \circ l$ is a loop in X based at v. Pick a path l joining distinct components of $f^{-1}(v)$. We construct a space K' from K by attaching a 1-cell e to ∂l and then attaching a 2-cell B to $e \cup l$. We would like to extend $f : K \to X$ to a map $f' : K' \to X$ such that $f'(e) = v$ and $f'^{-1}(v)$ does not meet the interior of the 2-cell B. If this can be done, then $f'^{-1}(v) = f^{-1}(v) \cup e$ which is a forest with $\alpha - 1$ components. Also f' represents ϕ, as f' extends f and K' deformation retracts to K. Hence f' has all the required properties. We will be able to construct such an extension f' if the loop $f \circ l$ has image in X_1 (or X_2) and is contractible in X. For then $f \circ l$ is null homotopic in X_1, and f' restricted to B is essentially this null homotopy. Our aim is to show that such a l exists.

Choose two distinct components A and B of $f^{-1}(v)$ and let L be a path in K from A to B. As $f_* : \pi_1(K) \to \pi_1(X)$ is onto, there is a loop γ in K based at L(0) such that $f \circ \gamma$ is homotopic to the loop $f \circ L$. Let l be the path $\gamma^{-1}L$ in K. This is a path in K joining A to B such that $f \circ l$ is a contractible loop in X.

We can suppose that l is a cellular map $I \to K$ by subdividing I and by choice of l. Thus we can express l as a union of subpaths $l_1, \ldots, l_n$ such that the ends of l_i lie in $f^{-1}(v)$ and $f \circ l_i$ is a loop in X_1 or X_2. Further we can suppose that the maps $f \circ l_i$ alternate between X_1 and X_2. (Note that l_i may meet components of $f^{-1}(v)$ in its interior.) We say that l has length n.

Let g_i denote the homotopy class of $f \circ l_i$ in $\pi_1(X, v)$. Suppose that some l_i has the two properties that g_i is trivial and that the endpoints of l_i lie in one component of $f^{-1}(v)$. Then we can alter l to l' by removing l_i and replacing it with a path l_i' in $f^{-1}(v)$ which joins the endpoints of l_i. Clearly l' has length less than n. By repeating this process, we can arrange that l has no subarcs l_i with these two properties.

Now the equation $l = l_1 \ldots l_r$ gives rise to the equation $1 = g_1 g_2 \ldots g_r$ in $\pi_1(X)$. As $\pi_1(X) = G_1 * G_2$ and the g_i's lie alter-

nately in G_1 and G_2, we deduce that some g_i is trivial. The corresponding l_i joins distinct components of $f^{-1}(v)$ and has $f \circ l_i$ contractible. We can now construct the required space K' and map $f' : K' \to X$ as previously described.

3. SUBGROUPS AND COVERING SPACES

We will apply the theory of covering spaces to the problem of describing subgroups of amalgamated free products. First, we consider free products.

Suppose given a group $G = G_1 * G_2$. As before we construct a space X with fundamental group G by taking CW-complexes X_1, X_2 with $\pi_1(X_i) = G_i$ and joining the basepoints of X_1 and X_2 with an interval E. We take the midpoint v of E as the basepoint of X. Now suppose that H is a subgroup of G. Then H is the fundamental group of some connected covering space $\tilde{X}$ of X, with projection map $p : \tilde{X} \to X$. Inside $\tilde{X}$, we have $p^{-1}(X_1)$ which is a covering space of X_1 and so consists of various connected covering spaces of X_1. Also $p^{-1}(X_2)$ is a union of connected covering spaces of X_2. Finally, as E is simply connected, $p^{-1}(E)$ is a union of copies of E. Thus $\tilde{X}$ looks like (and agrees up to homotopy with) a graph Γ with a covering space of X_1 or X_2 at each vertex. If Γ were a tree, then H would be the free product of the fundamental groups H_λ of all the spaces at the vertices of Γ. In general, Γ consists of a tree T with extra edges attached to T, where T is a maximal tree in Γ. Thus H will be the free product of all the groups H_λ and of a free group whose generators correspond to the edges of $\Gamma - T$.

Let $\tilde{v}$ denote the basepoint of $\tilde{X}$ and recall that $H = p_*(\pi_1(\tilde{X}, \tilde{v}))$. Let C be a component of $p^{-1}(X_1)$ and join it to $\tilde{v}$ by a path in $\tilde{X}$. We see that $p_*(\pi_1(C, \tilde{v}))$ is a conjugate of some subgroup of G_1. Thus the above description of a typical covering space of X leads at once to the following result.

Theorem 3.1 (Kuroš' subgroup theorem). *If H is a subgroup of $G = G_1 * G_2$, then H is the free product of a free group with subgroups of conjugates of G_1 or G_2.*

Corollary 3.2. If H is a subgroup of a free group, then H is free.

Corollary 3.3. If H is indecomposable and not infinite cyclic, and if $H \subset G_1 * G_2$, then H lies in a conjugate of G_1 or G_2.

Examples of an application of Corollary 3.3 would be when H is finite or abelian.

Exercise. Prove that a non-trivial direct product cannot be a non-trivial free product.

Lemma 3.4. If $G = G_1 * G_2$ and if $w^{-1}G_1 w \cap G_i$ is non-trivial, then $i = 1$, $w \in G_1$ and so $w^{-1}G_1 w \cap G_i = G_1$.

Proof. Clearly G_1 must be non-trivial, and we may as well suppose that G_2 is non-trivial. Now let g be a non-trivial element of G_1 such that $w^{-1}gw \in G_i$. We can write $w = \alpha w_1$, where $\alpha \in G_1$ and w_1 is a reduced word in G beginning in G_2. Thus $w^{-1}gw = w_1^{-1}(\alpha^{-1}g\alpha)w_1 = w_1^{-1}g'w_1$ where g' is a non-trivial element of G_1. Thus $w_1^{-1}g'w_1$ is a reduced word. But this is an element of G_i and so has length 1. Hence w_1 is trivial and so w lies in G_1. Hence $w^{-1}G_1 w = G_1$ and we have $G_1 \cap G_i$ non-trivial. This can only happen when $i = 1$, which completes the proof of the lemma.

Theorem 3.5. If G is a finitely generated group, then $G = G_1 * \ldots * G_n$, where each G_i is indecomposable. If also $G = G_1 * \ldots * G_n = H_1 * \ldots * H_m$ where each G_i and H_j is non-trivial and indecomposable, then $m = n$ and, by re-ordering, we have $G_i \cong H_i$ for each i. Further, for each i with G_i not infinite cyclic we have G_i conjugate to H_i.

Proof. The first sentence is just Corollary 2.2 stated again.

Now suppose that $G = G_1 * \ldots * G_n = H_1 * \ldots * H_m$, where each G_i, H_j is indecomposable. If each G_i is infinite cyclic, then G is free and Corollary 3.2 tells us that each H_j is free. As H_j is indecomposable it must be infinite cyclic and it now follows easily by abelianising and

using the basis theorem for f. g. abelian groups that $m = n$. Otherwise we can re-order the G_i's so that $G_1, \ldots, G_r$ are not infinite cyclic and $G_{r+1}, \ldots, G_n$ are infinite cyclic.

Corollary 3. 3 applied to $G_1 \subset H_1 * \ldots * H_m$ shows that, by re-ordering the H's, we have $u^{-1}G_1 u \subset H_1$ for some u in G. Hence H_1 is not infinite cyclic and Corollary 3. 3 applied to $H_1 \subset G_1 * \ldots * G_n$ shows that $v^{-1}H_1 v \subset G_i$ for some i. Hence $w^{-1}G_1 w \subset G_i$, where $w = uv$. Now Lemma 3. 4 shows that $w \in G_1$ and $i = 1$. Thus we have

$$G_1 = w^{-1}G_1 w \subset v^{-1}H_1 v \subset G_1 .$$

It follows that H_1 is conjugate to G_1 in G and hence also that H_1 is isomorphic to G_1.

Repeat this process for $G_2, \ldots, G_r$ to show that G_i is conjugate to H_i for $i = 1, 2, \ldots, r$. (Note that we cannot find two different G_i's conjugate to the same H_j as different G_i's cannot be conjugate to each other, by Lemma 3. 4.)

Consider $G/\langle G_1 * \ldots * G_r \rangle$, where $\langle X \rangle$ denotes the normal closure of a subset X of G. As G_i is conjugate to H_i for $i = 1, 2, \ldots, r$ we have

$$G_{r+1} * \ldots * G_n \cong G/\langle G_1 * \ldots * G_r \rangle \cong G/\langle H_1 * \ldots * H_r \rangle \cong H_{r+1} * \ldots * H_m .$$

The left hand group is free, and so each H_i, $i = r + 1, \ldots, m$ must be infinite cyclic. It now follows that $m = n$ and we have completed the proof of Theorem 3. 5.

One might ask whether an analogue of Theorem 3. 5 holds for non-f. g. groups. At the end of this chapter, we give an example which shows that the first part of the theorem fails for non-f. g. groups. However, the uniqueness result which is the second part of Theorem 3. 5 clearly applies to all groups which can be expressed as a finite free product of indecomposables.

The next step is to consider the structure of subgroups of amalgamated free products. Let us consider a group $G = A *_C B$. Let X_0 be a CW-complex with fundamental group A, let X_1 be a CW-complex with fundamental group B and let X_2 be a 2-dimensional CW-complex with fundamental group C. Lemma 1. 5 tells us that there are maps

$f_0 : X_2 \to X_0$ and $f_1 : X_2 \to X_1$ such that the induced maps of fundamental groups are the inclusions of C in A and B respectively. (By subdividing, we can suppose that f_0 and f_1 are cellular.) We construct a space X with fundamental group G by taking X_0, X_1 and $X_2 \times I$ and gluing $X_2 \times \{i\}$ to X_i using f_i, for $i = 0, 1$. We identify X_2 with the subspace $X_2 \times \{\frac{1}{2}\}$ of X.

Let H be a subgroup of G and let $\tilde{X}$ be the corresponding covering space of X with projection map $p : \tilde{X} \to X$. As before $p^{-1}(X_i)$ consists of a collection of connected covering spaces of X_i, for $i = 0, 1$ or 2. Thus $\tilde{X}$ is constructed from a collection of connected covering spaces of X_0 and X_1 and a collection of connected spaces of the form $Y \times I$, where Y is a covering space of X_2, by gluing $Y \times \{i\}$ to a covering space of X_i, for $i = 0, 1$. $\tilde{X}$ looks like a graph Γ with a space at each vertex and a (space $\times$ I) along each edge. If Γ were a tree, then H would be a multiple amalgamated free product where each amalgamation is of the type $A *_C B$ and not $A *_C$. In general, Γ is a tree T, with extra edges attached, and then H is a multiple amalgamated free product together with HNN extensions. Note that the form of H one obtains depends on the choice of a maximal tree T in Γ.

If $G = A *_C$ one can obtain a similar description of subgroups of H. We take a CW-complex X_0 with fundamental group A and a 2-dimensional CW-complex X_2 with fundamental group C and construct X from X_0 and $X_2 \times I$ by gluing $X_2 \times \partial I$ to X_0 appropriately.

We now introduce the terminology, due to Serre, of a graph of groups to describe the above sort of structure in a group.

Note that the word graph means a 1-dimensional CW-complex, so that a graph Γ may contain a loop i. e. an edge with its two endpoints identified. This gives rise to difficulties with orientations of such an edge. In order to avoid these difficulties we first introduce the idea of an abstract graph. Essentially this has twice as many edges as Γ, one for each orientation of an edge of Γ.

Definition. An abstract graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$, called the edges and vertices of Γ, an involution on $E(\Gamma)$ which sends e to $\bar{e}$, where $\bar{e} \neq e$, and a map $\partial_0 : E(\Gamma) \to V(\Gamma)$.

We define $\partial_1 e = \partial_0 \bar{e}$ and say that e joins $\partial_0 e$ to $\partial_1 e$.

An abstract graph Γ has an obvious geometric realisation $|\Gamma|$ with vertices $V(\Gamma)$ and edges corresponding to pairs $(e, \bar{e})$. When we say that Γ is connected or has some other topological property, we shall mean that the realisation of Γ has the appropriate property. An orientation of an abstract graph is a choice of one edge out of each pair $(e, \bar{e})$.

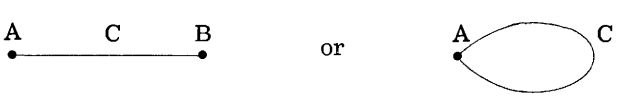
A graph of groups consists of an abstract graph Γ (which will always be tacitly assumed to be connected) together with a function $\mathcal{G}$ assigning to each vertex v of Γ a group G_v and to each edge e a group G_e, with $G_{\bar{e}} = G_e$, and an injective homomorphism $f_e : G_e \to G_{\partial_0 e}$. One may think of Γ as a partial category and $\mathcal{G}$ as a sort of functor. Similarly we may define a graph $\mathcal{X}$ of topological spaces, or of spaces with preferred basepoint: here it is not necessary for the map $X_e \to X_{\partial_0 e}$ to be injective, as we can use the mapping cylinder construction to replace the maps by inclusions, and this does not alter the total space defined below But we will suppose for convenience that the spaces are CW complexes and maps cellular.

Given a graph $\mathcal{X}$ of spaces, we can define a total space X_Γ as the quotient of $\cup\{X_v : v \in V(\Gamma)\} \cup \cup\{X_e \times I : e \in E(\Gamma)\}$ by the identifications

$$X_e \times I \to X_{\bar{e}} \times I \text{ by } (x, t) \to (x, 1 - t)$$

$$X_e \times 0 \to X_{\partial_0 e} \text{ by } (x, 0) \to f_e(x).$$

If $\mathcal{X}$ is a graph of (connected) based spaces, then by taking fundamental groups we obtain a graph $\mathcal{G}$ of groups (with the same underlying abstract graph Γ). The fundamental group G_Γ of the graph $\mathcal{G}$ of groups is defined to be the fundamental group of the total space X_Γ. Observe that in the cases when Γ has just one pair $(e, \bar{e})$ of edges



we obtain the products $A *_C B$, $A *_C$ already discussed, as follows by van Kampen's theorem (1.1 and 1.2). The general case may be considered

as derived by an iterated application of these constructions; however if it is treated in this way, the underlying geometry is liable to be obscured by computational complexities.

We now show that G_Γ does not depend on the choice of $\mathcal{X}$. First, for any $\mathcal{G}$ we can choose (using presentations) connected 2-dimensional CW complexes with $\pi_1(X_v, *) \cong G_v$ and $\pi_1(X_e, *) \cong G_e$. By Lemma 1.5, the homomorphisms f_e are induced by continuous maps $(X_e, *) \to (X_{\partial_0 e}, *)$. This defines a graph $\mathcal{X}$ of connected based spaces giving rise to $^0\mathcal{G}$. Next for any $\mathcal{X}$ we can attach cells of dimension ≥ 3 to each X_v, X_e to obtain aspherical spaces K_v, K_e still with the same fundamental group. Now the map $f_e : X_e \to X_{\partial_0 e}$ extends to a map $k_e : K_e \to K_{\partial_0 e}$ (there is no obstruction) so we have a new graph $\mathcal{K}$ of spaces, still inducing $\mathcal{G}$; its total space K_Γ is obtained from X_Γ by adding cells of dimension ≥ 3, so has the same fundamental group. But K_v is a space of type $(G_v, 1)$: its homotopy type is entirely determined by G_v (similarly K_e, G_e). Also the map k_e is determined up to homotopy by $f_e : G_e \to G_{\partial_0 e}$. Thus K_Γ is determined up to homotopy, and its fundamental group is unique up to isomorphism.

In order to relate the topology more closely to the group theory, we have insisted above on preservation of base points. However if the attaching maps $X_e \to X_{\partial_0 e}$ are altered by any homotopy (not necessarily base point preserving), the homotopy type and hence the fundamental group of X_Γ are unchanged. If the base point is pulled round a loop defining $g \in \pi_1(X_{\partial_0 e}, *)$, the homomorphism $G_e \to G_{\partial_0 e}$ is changed by conjugation by g. Thus even such a change will not alter G_Γ.

Corresponding to and generalising (1.6) and (1.7) we now have the

Proposition 3.6. (i) *If $\mathcal{G}$ is a graph of groups as above, each map $G_v \to G_\Gamma$ is injective.*

(ii) *If $\mathcal{K}$ is a graph of aspherical spaces as above, the total space K_Γ is aspherical.*

Proof. We start from the graph $\mathcal{K}$ of spaces. Observe that for each vertex v of Γ, the space

$$L_v = K_v \cup \bigcup_{\partial_0 e = v} (K_e \times I)$$

admits K_v as deformation retract, so its universal cover $\tilde{L}_v$ is contractible. Moreover, as each map $G_e \to G_v$ is injective, $\tilde{L}_v$ is obtained from $\tilde{K}_v$ by attaching copies of $\tilde{K}_e \times I$ with $\tilde{K}_e$ the universal cover of K_e, hence also contractible.

Now construct a space $Y = \cup Y_n$ by induction. Choose any vertex v_0 of Γ and set $Y_0 = \tilde{L}_{v_0}$. Now for any $n \geq 1$, in forming Y_{n-1} a number of copies of $\tilde{K}_e \times I$ will have been attached (each along $\tilde{K}_e \times 0$), for various edges e. We define Y_n to be the union of Y_{n-1} with a copy of $\tilde{L}_{\partial_1 e}$ for each such copy of $\tilde{K}_e \times I$, identified along $\tilde{K}_e \times I$. Since we are attaching contractible sets along contractible subsets, each Y_n is contractible.

Set $Y = \cup Y_n$ with the weak topology. Then Y also is contractible. There is an evident projection $Y \to K_\Gamma$; by construction K_Γ is evenly covered by Y. This proves (ii), and (i) follows since for each $K_v \subset K_\Gamma$, the induced covering of K_v contains the universal covering.

Remark. Assertion (ii) is equivalent to the exact sequences of Chiswell [30]. A normal form, in the style of Theorems 1.6 and 1.7, is given by Higgins [32].

We can now state our first result about subgroups of amalgamated free products.

Theorem 3.7. *If $G = A *_C B$ or $A *_C$ and if $H \subset G$, then H is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of A or B and the edge groups are subgroups of conjugates of C.*

Remarks. This result has an obvious generalization to the case where G is the fundamental group of a graph Γ of groups. The theorem covers the special case when the geometric realisation of Γ has a single edge.

There is a corollary of this result which is analogous to Corollary 3.3 in the case of free products. We say that a group G *splits over a*

subgroup C if $G = A *_C$ or $G = A *_C B$ with $A \neq C \neq B$. If G splits over some subgroup, we say that G is splittable. Note that $\mathbb{Z}$ is splittable as $\mathbb{Z} = \{1\} *_{\{1\}}$.

Corollary 3.8. If $G = A *_C B$ or $A *_C$ and if H is a finitely generated non-splittable subgroup of G, then H lies in a conjugate of A or B.

Proof. We know that H is the fundamental group of a graph Γ of groups. As H is finitely generated, there is a finite subgraph Γ' whose fundamental group equals H. Now the fact that H is not splittable implies that one of the vertex groups of Γ' is equal to H. The result follows.

Remark. The finite generation of H allows one to deduce that some vertex group of Γ equals H. If we consider non-finitely generated subgroups H, we see that H need not lie in a conjugate of A or B. For example, consider $G = \mathbb{Z} *_{\mathbb{Z}}$ where the two inclusion maps are the identity and multiplication by 2. Thus G has presentation $\{a, t : t^{-1}at=a^2\}$ The subgroup H of G generated by $t^n a t^{-n}$ for all integers n is isomorphic to the dyadic rationals and is therefore non-splittable, but of course H cannot be contained in an infinite cyclic group.

We must now consider covering spaces more closely. Recall paragraph 2 on page 138 of these notes. Let (X, x_0) be a based connected space with fundamental group G. Let H be a subgroup of G and let $(\tilde{X}, \tilde{x}_0)$ be the corresponding connected covering space, with projection map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$. Thus $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Lemma 3.9. There is a natural bijection $\bar{\phi} : H\backslash G \to p^{-1}(x_0)$, where $H\backslash G$ denotes the quotient of G under the action of H by left multiplication

Remark. The path lifting property of covering maps can be used to define bijections $p^{-1}(x_0) \to p^{-1}(x)$ for each $x \in X$.

Proof. First we define a map $\phi : G \to p^{-1}(x_0)$. Given $g \in G$, choose a map $(I, \partial I) \to (X, x_0)$ representing g. We will call such a map

a loop. Let l be the lift of this map starting at $\tilde{x}_0$ and define $\phi(g) = l(1) \in p^{-1}(x_0)$. This definition is independent of the choice of the loop chosen to represent g.

The map ϕ is a surjection. For given $y \in p^{-1}(x_0)$, choose a path l in $\tilde{X}$ from x_0 to y. Then $p \circ l$ is a loop in X representing some element $g \in \pi_1(X, x_0)$ and $\phi(g) = y$.

If $\phi(g_1) = \phi(g_2)$, then $l_1 l_2^{-1}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$, where l_i is a lift of a loop in X representing g_i. Thus $p \circ (l_1 l_2^{-1})$ represents an element h of H and we have the equation $g_1 g_2^{-1} = h$. Conversely if $g_1 g_2^{-1} = h \in H$, then $g_1 = hg_2$ and it is clear that $\phi(g_1) = \phi(g_2)$. Thus $\phi(g_1) = \phi(g_2)$ if and only if $g_1 g_2^{-1} \in H$ and so ϕ induces a bijection $\bar{\phi} : H\backslash G \to p^{-1}(x_0)$.

Lemma 3.10. If H is a normal subgroup of G, then G/H acts on $\tilde{X}$ by covering homeomorphisms with quotient X.

Proof. Let $g \in G$ and let $y \in p^{-1}(x_0)$ be the point determined by g. Then $p_*(\pi_1(\tilde{X}, y)) = g^{-1}Hg$ which equals H as H is normal in G. The uniqueness of covering spaces corresponding to H shows that there is a unique covering homeomorphism $\psi_g : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, y)$. I claim that this process defines a homomorphism of G to the group of covering homeo morphisms of $\tilde{X}$. One need only show that $\psi_{g_1 g_2}(\tilde{x}_0) = \psi_{g_1} \circ \psi_{g_2}(\tilde{x}_0)$, as two covering homeomorphisms which agree on $\tilde{x}_0$ must be equal by the uniqueness result again. Let l_1 and l_2 be paths in $\tilde{X}$ starting at $\tilde{x}_0$ which are lifts of loops in X representing g_1 and g_2 respectively. Then $\psi_{g_1} \circ l_2$ is a path in $\tilde{X}$ starting at $\psi_{g_1}(\tilde{x}_0)$ and still lifting a loop in X representing g_2. Thus $\psi_{g_1} \circ l_2$ begins where l_1 ends and we deduce that $\psi_{g_1 g_2}(\tilde{x}_0) = \psi_{g_1} \circ l_2(1) = \psi_{g_1} \circ \psi_{g_2}(\tilde{x}_0)$ as required. It is clear that the kernel of this homomorphism is H, so that we do have an action of G/H on $\tilde{X}$.

The quotient of $\tilde{X}$ by the action of G/H has a natural projection π to X and π is a covering map. Also, for each $x \in X$, $\pi^{-1}(x)$ is a single point as $\pi^{-1}(x_0)$ is a single point. Hence π is a homeomorphism and this completes the proof of the lemma.

Before going further, we give an application of this result.

Theorem 3.11. If $G = A * B$ where A, B are non-trivial and H is a finitely generated, normal subgroup of G, then H is trivial or has finite index in G.

Proof. We suppose that H has infinite index in G and will prove that H must be trivial. We know that H is the fundamental group of a graph Γ of groups, where the edge groups are trivial and the vertex groups H_λ are subgroups of conjugates of A or B. If T is a maximal tree in Γ, then $H = F * (* H_\lambda)$ where F is a free group whose generators correspond to the edges of $\Gamma - T$. The fact that H is normal in G tells us that G/H acts on Γ with quotient an interval.

As H has infinite index in G, we deduce that Γ has infinitely many edges. As H is finitely generated, we deduce that $\Gamma - T$ is finite and only finitely many of the groups H_λ are non-trivial. Thus H is the fundamental group of some connected finite subgraph Γ' of Γ. Let E be an edge of $\Gamma - \Gamma'$. Then removing E from Γ gives two subgraphs Γ_1 and Γ_2 one of which has trivial fundamental group. As G/H acts transitively on the edges of Γ, we deduce that every edge of Γ has these properties. Thus Γ must be a tree and at most one vertex group can be non-trivial. Thus H is contained in a conjugate of A or B. As H is normal in G, it must lie in the intersection of all conjugates of A (or of B). But $A \cap b^{-1}Ab$ is trivial for any non-trivial element $b \in B$. Hence H is trivial.

Q. E. D.

Exercise. Is there an analogous result when $G = A *_C B$ or $A *_C$?

We now return to covering spaces. The aim of our next result is to give a more precise structure theorem for subgroups of amalgamated free products.

Let H be a subgroup of $G = \pi_1(X, x_0)$, and let $\tilde{X}$, $\tilde{x}_0$ be as before. Let Y be a subspace of X which contains x_0, such that inclusion of Y in X induces an injective map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$. We denote the image group by A and identify $\pi_1(Y)$ with this subgroup of G.

Lemma 3.12. *There is a natural correspondence* $\overline{\theta}$ *between the double cosets* HgA *and the components of* $p^{-1}(Y)$ *in* $\tilde{X}$.

Proof. Let i denote the inclusion of $p^{-1}(x_0)$ in $p^{-1}(Y)$ and recall the map $\phi : G \to p^{-1}(x_0)$. We define $\theta : G \to p^{-1}(Y)$ by $\theta = i \circ \phi$.

Suppose that $\theta(g_1)$ and $\theta(g_2)$ lie in the same component of $p^{-1}(Y)$. Then they can be joined by a path l in $p^{-1}(Y)$. Let $\alpha \in A$ be the element of $\pi_1(X, x_0)$ represented by the loop $p \circ l$. Then, by projecting into X, we see that $g_1 \alpha g_2^{-1} \in H$, so that $g_2 = hg_1\alpha$ for some $h \in H$. Conversely, if $g_2 = hg_1\alpha$ for some elements $\alpha \in A$ and $h \in H$, then lifting to $\tilde{X}$ tells us that $\theta(g_1)$ and $\theta(g_2)$ can be joined by a path l in $p^{-1}(Y)$, where l is a lift of α.

Hence θ induces the required bijection $\overline{\theta}$.

Lemma 3.13. *Let* $g \in G$, $y = \phi(g) \in p^{-1}(x_0)$ *and let* C *be the component of* $p^{-1}(Y)$ *which contains* y. *Let* λ *be a loop in* X *representing* g *and let* l *be the lift of* λ *which goes from* $\tilde{x}_0$ *to* y. *Then* $p_*(\pi_1(C, \tilde{x}_0)) = H \cap gAg^{-1}$, *where we define* $\pi_1(C, \tilde{x}_0)$ *by using the path* l.

Proof. We know that $p_*(\pi_1(C, y)) \subset A$, and so $p_*(\pi_1(C, \tilde{x}_0)) \subset gAg^{-1}$ As $p_*(\pi_1(C, \tilde{x}_0)) \subset H$, we have $p_*(\pi_1(C, \tilde{x}_0)) \subset H \cap gAg^{-1}$.

Consider an element $\beta = g\alpha g^{-1}$ of $H \cap gAg^{-1}$, where $\alpha \in A$. Let μ be a loop in Y representing α. Then $\lambda\mu\lambda^{-1}$ is a loop in X representing β. We know that $\lambda\mu\lambda^{-1}$ lifts to a loop in $\tilde{X}$, because $\beta \in H$. Thus λ lifts to l and λ^{-1} lifts to l^{-1} and so $\lambda\mu\lambda^{-1}$ lifts to $l m l^{-1}$ where m is some loop in $p^{-1}(Y)$ based at y. Therefore β lies in $p_*(\pi_1(C, \tilde{x}_0))$ and we have shown that $p_*(\pi_1(C, \tilde{x}_0)) = H \cap gAg^{-1}$ as required.

We can now state a more precise version of the subgroup theorem. Similar statements can be found in [4], [15], [22], [28]. For convenience in the case $G = A *_C$, where we have two injections i_1 and i_2 of C into A, we identify C with $i_1(C)$. Then $i_2(C) = t^{-1}Ct$, and the subgroup $C = i_1(C)$ of A is also a subgroup of tAt^{-1}.

Subgroup Theorem 3.14. If H is a subgroup of $G = A *_C$ (or $A *_C B$), then H is the fundamental group of a graph Γ of groups. The vertices of Γ correspond to the double cosets HgA (and HgB), and the corresponding groups are $H \cap gAg^{-1}$ (and $H \cap gBg^{-1}$). The edges of Γ correspond to the double cosets HgC and the corresponding groups are $H \cap gCg^{-1}$.

If $G = A *_C B$, the two ends of the edge HgC are the vertices HgA and HgB and the injections of the associated groups are simply the inclusion mappings.

If $G = A *_C$, the two ends of the edge HgC are the vertices HgA and HgtA. The injections of the associated groups are simply the inclusion mappings.

A corresponding theorem for subgroups H of G, where G is the fundamental group of any graph of groups, follows from the same lemmas. We leave the precise formulation to the reader.

As applications of the subgroup theorem, we prove the following results.

Lemma 3.15. If $G = G_1 *_C G_2$, then either $gG_1g^{-1} \cap G_i$ is a subgroup of a conjugate of C, or $i = 1$ and $g \in G_1$, so that $gG_1g^{-1} \cap G_i = G_1$.

Proof. Let $H = gG_1g^{-1} \cap G_i$. Then H is the fundamental group of a graph Γ of groups. The vertices v_1, v_2 of Γ corresponding to the double cosets HgG_1 and HG_i have associated groups $H \cap gG_1g^{-1} = H$ and $H \cap G_i = H$. If v_1 and v_2 are distinct vertices of Γ, choose a path in Γ joining them. As the inclusion of each of the groups associated to v_1, v_2 is an isomorphism with H, we see that each vertex and edge of this path has associated group H. Thus H is the group associated to some edge and so lies in some conjugate of C. The only other possibility is that $v_1 = v_2$. This implies that $i = 1$ and $HgG_1 = HG_1$. Thus $g \in G_1$, and the result is proved.

Next we give the promised example of a non-f. g. group G which fails to satisfy the conclusion of Theorem 3.5. This example is due to Kuroš [17].

Example. The group $G = \{a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \mid a_{n-1} = [a_n, b_n], \forall n \geq 1\}$ cannot be expressed as a free product of indecomposable subgroups.

First observe that $G = \{a_1, b_1\} *_{C_1} \{a_2, b_2\} *_{C_2} \ldots$, where each factor group is free of rank 2, each C_i is infinite cyclic, the inclusion map $C_i \to \{a_i, b_i\}$ sends a generator to a_i and the inclusion map $C_i \to \{a_{i+1}, b_{i+1}\}$ sends the same generator to $[a_{i+1}, b_{i+1}]$.

Next observe that

$$G = \{b_1\} * \{a_1, a_2, \ldots, b_2, b_3, \ldots \mid a_{n-1} = [a_n, b_n], \forall n \geq 2\} \cong \mathbb{Z} * G.$$

Hence G can be expressed as a free product involving any given (finite) number of factors. Hence G cannot be a free product of n indecomposables, for any integer n, as the proof of the uniqueness result of Theorem 3.5 would apply to show that any factorisation of G has no more than n factors - a contradiction. Hence if G can be expressed as a free product of indecomposables, then G must have an infinite number of factors. The last step is to show that this also is impossible.

Suppose that $G = G_1 * G_2 * \ldots$, where each G_i is indecomposable and non-trivial. Consider the element a_0 of G. For some n, a_0 must lie in $G_1 * \ldots * G_n$, which we denote by A. Thus $G = A * B$, with A and B non-trivial and $a_0 \in A$. Our decomposition of G as an infinite amalgamated free product shows that each a_i is a non-trivial element of G. The fact that a_0 lies in A shows that a_1, b_1 lie in A, by Lemma 3.16 below. As $a_1 \in A$, we see a_2, $b_2 \in A$ by the same lemma. By repeating this argument, we see that $G \subset A$, contradicting the hypothesis that B is non-trivial. This contradiction proves the required result.

Lemma 3.16. *If a group* $G = A * B$, $A \neq \{1\} \neq B$, *and if* $g = [g_1, g_2]$ *is a non-trivial element of* A, *then* g_1 *and* g_2 *lie in* A.

Proof. Let H be the subgroup of G generated by g_1, g_2. By the Subgroup Theorem, $H = (H \cap A) * C$, for some subgroup C of H. We must prove that $H \subset A$. If this is not the case, then each of $H \cap A$ and

C is non-trivial and so Corollary 2.1 of Grusko's Theorem shows that each group is cyclic. Hence the abelianisation homomorphism $H \to H/H'$ injects $H \cap A$ and C. This contradicts the fact that g is a non-trivial commutator in $H \cap A$. Hence H must be contained in A.

We finish this section by proving the famous embedding theorem of Higman and Neumann [13] which states that any countable group can be embedded in a 2-generator group. A nice example to consider is the subgroup K of F_2, the free group on a and b, which is the kernel of the homomorphism $F_2 \to \mathbb{Z}$, which sends b to a generator and a to the identity. Let X denote the wedge of 2 circles, so that $\pi_1(X) \cong F_2$. The covering space $\tilde{X}$ of X corresponding to K consists of a copy of R together with a circle attached at each integer point. Thus K has basis $\{b^{-n}ab^n : n \in \mathbb{Z}\}$. If one started with a countably generated free group $K = \{x_n : n \in \mathbb{Z}\}$, one would embed it in a finitely generated group by adding an element b to K which makes all the x_i's conjugate. More precisely, one has the shift automorphism of K sending x_n to x_{n+1}, for each n, and one takes the extension of K by $\mathbb{Z}$ determined by this automorphism. The new group generated by x_0 and b is, of course, isomorphic to F_2. The idea of the proof of the embedding result is to do the same sort of thing in general, i.e. make lots of generators conjugate.

Theorem 3.17. If G is a countable group, then G can be embedded in a 2-generator group.

Proof. Let $x_1, x_2, \ldots$ be a generating set for G. We embed G in $G_1 = G * \mathbb{Z}$. Let t be a generator of $\mathbb{Z}$, and write $y_i = x_i t$, $y_0 = t$. Then G_1 is generated by $y_0, y_1, \ldots$ and each y_i has infinite order. Now let $G_2 = \{G_1, t_0, t_1, \ldots : t_i^{-1} y_i t = y_{i+1}\}$. Then G_2 is obtained from G_1 by an infinite sequence of HNN extensions and so $G_1 \subset G_2$. A set of generators for G_2 is $y_0, t_0, t_1, \ldots$. The subgroup K of G_2 generated by the t_i's is free and has the t_i's as a basis. To see this observe that by killing G_1 one obtains a homomorphism of G_2 to a free group F which maps the t_i's to a basis for F. Thus we can embed K in F_2 as in the discussion preceding the theorem and we let

$G_3 = G_2 *_K F_2$. Again $G_2 \subset G_3$ and G_3 is generated by y_0, a and b where a denotes one of the t_i's. Finally we observe that the subgroup H of G_3 generated by y_0 and b is free of rank two. This can be seen by applying the Subgroup Theorem, as $H \cap G_2$ and $H \cap F_2$ are infinite cyclic and $H \cap K$ is trivial. Thus we can construct $G_4 = G_3 *_{F_2}$ where our two inclusions of F_2 in G_3 have images H and F_2. We choose $G_4 = \{G_3, s : s^{-1}as = b, s^{-1}bs = y_0\}$. Then G_4 is generated by a and s which completes the proof of the embedding theorem.

4. GROUPS ACTING ON GRAPHS

The subject matter of this chapter is a reworking of the Bass-Serre theory [22]. We consider a (continuous) action of a group G on a (topological) graph Γ: clearly this corresponds also to an action on the corresponding abstract graph. We say that G acts without inversions if whenever an element g of G fixes an edge e of Γ, it fixes each point of e. In the abstract setting, this means that $g.e = \bar{e}$ is forbidden. Given any action, the process of subdividing each edge once by an extra vertex in the middle gives us an action without inversions.

The following simple example will be of use in Section 6. Let G be a group, $S \subset G$ a subset (not subgroup), $\Gamma = \Gamma(S, G)$ the (geometric) graph with vertex set G and, for each $(g, s) \in G \times S$ a single edge e(g, s) joining g to gs. There is an obvious action of G on Γ, where $h \in G$ takes the vertex g to the vertex hg; the edge e(g, s) to the edge e(hg, hgs). Only the identity element of G can leave a vertex or edge fixed, so G acts freely (without inversions). Note that even if $s^2 = 1$ we do not identify the edges e(g, s) and e(gs, s) with the same endpoints.

Proposition 4.1. (i) Γ contains no loop $\iff 1 \notin S$.
(ii) Γ is a simplicial complex $\iff S \cap S^{-1} = \emptyset$.
(iii) Γ is connected $\iff$ S generates G.
(iv) Γ is a tree $\iff$ S freely generates G.

Proof. (i) and (ii) are trivial.

If H denotes the subgroup generated by S, and Γ' the full sub-

graph on H, then Γ' is open (there are no edges with just one end in Γ') and connected (any word - e.g. $s_1 s_2 s_3^{-1} s_4$ - in $S \cup S^{-1}$ defines, in an obvious way, a path in Γ' joining 1 to the element in H it represents). Now (iii) follows.

Finally if Γ is a tree, it is contractible; as G acts freely on Γ, we have an isomorphism of G on $\pi_1(G\backslash\Gamma, *)$. But $G\backslash\Gamma$ is the graph with one vertex and edges labelled by S, so its fundamental group is the free group F(S). Moreover the composed isomorphism $F(S) \to G$ takes $s \in F(S)$ to the class of the loop 's'; lifting this, we get the edge of Γ from 1 to s, which corresponds to $s \in G$. This argument is reversible: if S freely generates G, then $G \cong F(S) \cong \pi_1(G\backslash\Gamma, *)$, so Γ is the universal cover of $G\backslash\Gamma$, hence contractible - i.e. a tree.

Now suppose given a graph $\mathcal{G}$ of groups, with vertices v and edges e corresponding to groups G_v, G_e with $G_{\bar{e}} = G_e$, and injections $\alpha(e) : G_e \to G_{\partial_0 e}$. As before, we choose a corresponding graph of connected spaces X_v, X_e with total space X_Γ, and fundamental group G_Γ. Since by (3.6) the natural maps $G_v \to G_\Gamma$, $G_e \to G_\Gamma$ are injective, the universal cover $\tilde{X}_\Gamma$ is a union of copies of the universal covers $\tilde{X}_v$, $\tilde{X}_e \times I$.

In $\tilde{X}_\Gamma$, identify each copy of $\tilde{X}_v$ to a point, and each copy of $\tilde{X}_e \times I$ to a copy of I, giving a quotient space Z with projection $\pi : \tilde{X}_\Gamma \to Z$. Clearly Z is a graph. We define $j : Z \to \tilde{X}_\Gamma$ by first choosing for each vertex (edge) of Z a point V(E) in the corresponding copy of $\tilde{X}_v(\tilde{X}_e)$. Then divide each edge of Z into three parts: j maps the middle part to $E \times I$, and the end parts to paths in the connected space $\tilde{X}_v$ joining the corresponding points E, V. Clearly $\pi \circ j$ is homotopic to the identity, so Z is connected and simply-connected, and hence is a tree.

The map π is compatible with the natural action of G_Γ on $\tilde{X}_\Gamma$, so we inherit an action of G_Γ on Z. This action has no inversions. The isotropy group of each vertex (obtained from collapsing a copy of $\tilde{X}_v$) is a conjugate of G_v, and of an edge (collapsed from $\tilde{X}_e \times I$) is a conjugate of G_e. As G_Γ acts without inversions, $G_\Gamma\backslash Z$ is also a graph and in fact coincides with the geometric realization $|\Gamma|$ of the original graph Γ: there is an obvious map onto $|\Gamma|$ and each vertex (edge) of $|\Gamma|$ deter-

mines $X_v(X_e \times I)$ in X_Γ, hence a collection of copies of $\tilde{X}_v(\tilde{X}_e \times I)$ in $\tilde{X}_\Gamma$, transitively permuted by G_Γ, hence a single vertex (edge) of $G_\Gamma \backslash Z$. Thus we recover the original graph $\mathcal{G}$ of groups from the action of G_Γ on Z. There is a slight problem here: how to choose the subgroups within their conjugacy classes to obtain the desired inclusions. This will be dealt with below.

We now start from an action of a group G on a tree Y, having no inversions, and show how to construct a graph of groups with fundamental group G. Choose a connected CW complex U with fundamental group G: then $\tilde{U}$ is simply connected and G acts freely on it, hence also (diagonally) on $\tilde{U} \times Y$. Consider the quotient X and the projection

$$X = G\backslash(\tilde{U} \times Y) \to G\backslash Y = \Gamma, \text{ say.}$$

Since G acts without inversions, Γ is a graph and each vertex (edge) of Y projects isomorphically onto one of Γ. For a vertex v (edge e) with isotropy group G_v (G_e) in G, we see that $G\backslash(\tilde{U} \times v)$ ($G\backslash(\tilde{U} \times e)$) has fundamental group G_v (G_e). Thus X has the structure of a graph $\mathfrak{X}$ of connected spaces realising a graph $\mathcal{G}$ of groups. Since G acts freely on the 1-connected space $(\tilde{U} \times Y)$, we have $G = \pi_1(X)$ the fundamental group of $\mathcal{G}$. Observe that U plays no essential role in the construction of $\mathcal{G}$, which could be expressed purely algebraically except for a certain vagueness about conjugates which will be considered below.

In order to deal with the points at the end of the two preceding paragraphs, and also to obtain a more precise formulation of the result which can be used for explicit calculation with words, we must now consider base points. The usual procedure with a CW-complex K is to choose a maximal tree T in the 1-skeleton $K^{(1)}$ (a graph). This is contractible and contains all the vertices. Hence $K \to K/T$ is a homotopy equivalence, and K/T a complex with only one vertex, which we take as base point. Thus the edges of K <u>not</u> in T give generators of $\pi_1(K)$.

Now let G act without inversions on a (connected) graph Y, so that $X = G\backslash Y$ is also a graph; let T be a tree in X containing the vertex v, and let $\tilde{v} \in Y$ lie over v.

Proposition 4.2. There is a lifting $j : T \to Y$ of the inclusion of T in X; moreover, we can take $j(v) = \tilde{v}$.

Proof. Applying Zorn's Lemma, we see that there is a maximal pair (T', j'): T' a subtree of T (containing v), $j' : T' \to Y$ over the inclusion (with $j'(v) = \tilde{v}$). If $T' \neq T$, let w be a vertex of $T - T'$; since T is connected, we can join w to v by an edge path, and at least one edge in the path, say e, has one vertex v_0 which is in T' and one which is not. Now e is the image of an edge $\tilde{e}$ of Y, one of whose vertices $\tilde{v}_0$ lies over v_0. As $\tilde{v}_0$ and $j'(v_0)$ lie over v_0, they are equivalent under G. If $g.\tilde{v}_0 = j'(v_0)$, we can extend j' over e by setting $j(e) = g.\tilde{e}$. This contradicts the supposed maximality and proves the result.

Returning now to the action of G on the tree Y, we choose a maximal tree T in $\Gamma = G\backslash Y$, a lifting $j : T \to Y$ with $j(T) = \tilde{T}$, and use these as 'extended base points'. Over each vertex v of Γ there is just one vertex $\tilde{v}$ of $\tilde{T}$: we define G_v as the stabiliser of $\tilde{v}$. If e is an edge of T, we have the edge $\tilde{e} = j(e)$, and call its stabiliser G_e. For each other edge e of Γ we choose an edge $\tilde{e}$ of Y over e with $\partial_0\tilde{e} = (\partial_0 e)^{\sim}$, and an element $g_e \in G$ such that $\partial_1\tilde{e} = g_e.(\partial_1 e)^{\sim}$, and define G_e to be the stabiliser of $\tilde{e}$. The map $\alpha_0(e)$ is the inclusion map; $\alpha_1(e)$ is induced by conjugation by g_e. Note that we have implicitly chosen e from the pair $(e, \bar{e})$ (one could set $\tilde{\bar{e}} = (g_0^{-1}\tilde{e})^{-}$, $g_{\bar{e}} = g_e^{-1}$, but then would have $G_{\bar{e}} \neq G_e$).

We have considered two constructions above. Given a graph $\mathcal{G}$ of groups, realised by a graph $\mathcal{X}$ of spaces, we defined a quotient Z of $\tilde{X}_\Gamma$, proved it a tree, and obtained an action of G_Γ on it. Conversely, given an action (all actions supposed without inversions) of a group G on a tree Y we defined (following (4.2)), using certain choices (maximal tree T, liftings $\tilde{T}$, $\tilde{e}$, elements g_e) a graph of groups over $\Gamma = G\backslash Y$. The key result of the theory is

Theorem 4.3. These two constructions are mutually inverse up to isomorphism and (for graphs of groups) replacing the $\alpha_i(e)$ by conjugate homomorphisms.

Proof. Most of the proof was given above. Starting from the action of G on Y, the graph of groups over $\Gamma = G\backslash Y$ is realised by a graph of spaces with total space $G\backslash(\tilde{U} \times Y)$, U a connected CW complex with fundamental group G. The collapsing process on the universal cover $(\tilde{U} \times Y)$ gives back the original tree Y and action of G on it.

Now suppose given a graph $\mathcal{G}$ of groups and construct (as above) an action of the fundamental group G_Γ on a tree Z. We have already observed that the isotropy groups of vertices and edges agree up to conjugacy with the images in G_Γ of the given groups G_v, G_e. It remains to identify the injections $\alpha_i(e)$, where more care is necessary.

Take $\mathcal{X}$ as a graph of based spaces, so we can identify Γ with a subset of X, and choose a maximal tree T in Γ. Since T is contractible, we can lift it to $\tilde{T} \subset \tilde{X}_\Gamma$. For each edge e of $\Gamma - T$ there is a unique lift $\tilde{e} \subset \tilde{X}_\Gamma$ with $\partial_0 \tilde{e} \in \tilde{T}$, and a unique $g_e \in G_\Gamma$ with $(g_e^{-1} . \partial_0 \tilde{e}) \in \tilde{T}$ Now $\mathcal{G}$ is isomorphic to a graph of groups in which each $\alpha_0(e)$, and those $\alpha_1(e)$ with $e \in T$, are inclusions; so we can identify each G_e, G_v with a subgroup of G_Γ.

Now $\pi : \tilde{X}_\Gamma \to Z$ maps $\tilde{T}$ isomorphically to a tree $\tilde{T}$ over T. Using the action of G_Γ on Z to define a graph of groups as above, we obtain the same subgroups G_e and G_v; each $\alpha_0(e)$ and those $\alpha_1(e)$ with $e \in T$ are inclusions. For $e \notin T$, $\alpha_1(e)$ is induced by conjugation by g_e. In the given graph of groups $\alpha_1(e)$ was induced by the map $f_e^1 : (X_e, *) \to (X_{\partial_1 e}, *)$. There is a unique path p in T joining $\partial_1 e$ to $\partial_0 e$; as we have identified G_e (via $X_e \to X_{\partial_0 e}$) with a subgroup of $G_{\partial_0 e}$, the map f_e^1 cannot be regarded as preserving base points, which have to be translated along the path p. Thus $\alpha_1(e)$ is induced by conjugation by the element g' of G_Γ represented by the closed path $p.e.$ Now p lifts to the path in $\tilde{T}$ joining $(\partial_1 e)\tilde{}$ to $(\partial_0 e)\tilde{} = \partial_0 \tilde{e}$, so the lift of $p.e.$ joins it to $\partial_1 \tilde{e}$. Thus $g' . (\partial_1 e)\tilde{} = (\partial_1 e)\tilde{}$, which identifies g' with g_e and hence concludes the proof.

Remark. The reader may already have observed that our two inverse constructions can be formulated in purely algebraic terms. We feel however that the above proof of the key theorem is more intuitive than any

involving cancellation arguments.

We now note some special cases of the theorem.

Corollary 4.4. Let $G\backslash Y$ consist of a single edge $v \overset{e}{\bullet\!\!-\!\!-\!\!\bullet} v'$. Then $G \cong G_v *_{H_e} G_{v'}$.

Corollary 4.5. Let $G\backslash Y$ consist of a single loop. Then Y contains the edge $\tilde{v} \overset{\tilde{e}}{\bullet\!\!-\!\!-\!\!\bullet} g.\tilde{v}$ and $G \cong G_v *_{H_e}$.

A somewhat different application arises from considering free actions. Of course, if G acts freely on a tree, it is free. The theorem allows us to write down a set of free generators. Suppose in particular that F is the free group on a set S, and G a subgroup of F. Write down the graph Γ_S for F: by (4.1) it is a tree, and the action of F on it induces a free action of G.

Identify the vertices of Γ_S with the corresponding elements of F. Then the path in the tree joining 1 to the element with reduced word $t_1 \dots t_n$ (where each $t_i \in S \cup S^{-1}$) goes through the successive vertices t_1, t_1t_2, $t_1t_2t_3$, $\dots$. Thus if $\tilde{T} \subset \Gamma_S$ contains 1, it contains the initial segments of the vertices of $\tilde{T}$.

Since the action of G is free, the lift of an edge of $G\backslash\Gamma_S = X$ is determined by its initial vertex. Thus the preferred lifts of the edges of $X - T$ are just those edges of Γ_S whose initial vertex is in T but terminal vertex is not. Applying the theorem we have

Proposition 4.6. The left cosets of G in F are canonically represented by the set R of vertices of $\tilde{T}$: if a reduced word $w = uv$ belongs to R, so does u. If $W = \{(t, s) \in R \times S : ts \notin R\}$ and for each $w = (t, s) \in W$ we write $ts = g_w u_w$ with $g_w \in G$, $u_w \in R$ then $\{g_w : w \in W\}$ is a free basis of G.

We conclude this section with brief mentions of two alternative approaches. The first follows a paper of Serre [23]. We say that G has property (FA) if for any action of G on a tree Y, there is a fixed point of G in Y.

Theorem 4.7. G has (FA) if and only if G is (i) unsplittable, and (ii) not a union of an increasing sequence of subgroups.

Note that for countable G, (ii) is equivalent to being finitely generated.

Proof. (FA) $\Rightarrow$ (i) by Corollaries 4.4 and 4.5: any decomposition induces an action without a fixed point. As to (ii), if $G = \cup G_n$ with $G_n \subset G_{n+1}$ we form a graph with vertices $\cup(G/G_n)$ and for each vertex gG_n an edge joining it to gG_{n+1}. It is immediate that this is a tree, and that the natural action of G on it has no fixed point.

Conversely if (i) and (ii) hold and G acts on Y, G is the universal group of the graph of groups $G\backslash Y$. By (ii), G is also the universal group of a finite subgraph. If this subgraph is not a tree, G is splittable; if the subgraph is a tree, we still have a splitting unless G coincides with one of the vertex groups - i. e. has a fixed point in Y.

Some interesting examples of the above are given in Serre's paper [23] and several more in his monograph [22, Chapter 6].

We conclude this paragraph by mentioning length functions. These were introduced by Lyndon [34] to permit inductive arguments: they constitute an axiomatic generalisation of the length of a reduced word as in 1.4, 1.6 or 1.7 above. It was shown by Chiswell [3] however that every function satisfying the axioms defines an action on a tree and hence comes from a decomposition of G as fundamental group of a graph of groups. Thus here we have a further equivalent concept.

5. ENDS

The definition of ends, and construction of the end point compactification (for a peripherally compact space) was achieved by Freudenthal in 1931 [9]; and the application to group theory initiated by himself [10], [11], Hopf [14] and Specker [24]. We present a somewhat simplified version, adapted to the present applications.

Let X be a locally finite simplicial complex. For each finite subcomplex K, the number of connected components of $X - K$ is finite; denote by $n(K)$ the number of infinite ones (equivalently, having noncompac closure in X). Now define the number of ends $e(X) = \sup n(K)$. Clearly $e(X) = 0 \Longleftrightarrow X$ is finite; otherwise $e(X)$ is a positive integer or $+\infty$.

If $X = \mathbb{R}$ and K is a point, clearly $n(K) = 2$. On the other hand, any compact K is contained in a closed interval $J : \mathbb{R} - J$ has only two components, and a component of $\mathbb{R} - K$ meeting neither is contained in J, hence finite. Thus $e(\mathbb{R}) = 2$. Similarly, since the complement of a (large) disc is connected, $e(\mathbb{R}^n) = 1$ for any $n \geq 2$.

As X is locally finite, for any finite K the (open) subcomplex $st(K)$ consisting of all simplices with a vertex in K is finite, and clearly $n(\overline{st\ K}) \geq n(st\ K) \geq n(K)$. Now any point of $X - (st\ K)$ can be joined by a path avoiding $st\ K$ to a vertex not in K, and if two such vertices can be joined by a path avoiding $st\ K$, as none of the vertices of the simplices met by the path are in K, we can find a path along edges not in $st\ K$. It follows that in computing $e(X)$ we may ignore all simplices of dimension > 1, and work in the 1-skeleton. This can now be formalized. The cochain complex $C^*(X)$ of X (coefficients $\mathbb{Z}_2$ = integers mod 2 understood contains a subcomplex $C_f^*(X)$ of cochains with finite support. Note: the fact that $C_f^*(X)$ is closed under the coboundary follows from local finiteness of X. Write $C_e^*(X)$ for the quotient complex, and $H_e^*(X)$, $H_f^*(X)$ for the cohomology groups of $C_e^*(X)$, $C_f^*(X)$. Then the short exact sequence $0 \to C_f^*(X) \to C^*(X) \to C_e^*(X) \to 0$ induces a long exact sequence of cohomology groups.

Our interest in these comes from

Proposition 5.1. $e(X)$ *is the dimension of* $H_e^0(X)$ *over* $\mathbb{Z}_2$.

Proof. Observe that $H_e^0(X) = \delta^{-1}(C_f^1(X))/C_f^0(X)$ is the quotient of 0-cochains with finite coboundary by finite 0-cochains. Also, by the above, we may suppose X 1-dimensional.

Now if the 0-cochains $c_1, \ldots, c_n$ define linearly independent elements of $H_e^0(X)$, as each δc_i is finite we can choose a finite subcomplex K containing the supports of all δc_i. But then for each edge e not in K, each c_i takes the same value at both ends of e. Thus for each connected component A of $X - K$, each c_i takes a constant value $c_i(A)$ on the vertices of A. If there were only $r < n$ infinite components A, there would be a nontrivial linear relation $\Sigma \lambda_i c_i(A) = 0$ holding for all such A. But then $\Sigma \lambda_i c_i$ would be a finite cochain, contradicting our

choice. Hence $n \leq \dim H_e^0(X)$ implies $e(X) \geq n$.

Conversely, if $e(X) \geq n$ we choose K finite with $n(K) \geq n$, and let $A_1, \ldots, A_n$ be distinct infinite components of $X - K$. Define the cochain c_i to take the value 1 on vertices of A_i, 0 on other vertices of X. Then if $\delta c_i(e) = 1$, e has one end in A_i, the other not (and hence in K), so e is one of the finitely many edges of st K. So each δc_i is finite and, by construction, the c_i are independent modulo finite cochains. Hence $\dim H_e^0(X) \geq n$.

We next construct another theory, analogous to the above. For any group G, let PG be the power set of all subsets. Under Boolean addition ('symmetric difference') this is an additive group of exponent 2. Write FG for the additive subgroup of finite subsets. Now define

$$QG = \{A \subset G : \forall g \in G,\ A + Ag \text{ is finite}\}.$$

We refer to two sets A and B whose difference lies in FG as almost equal, and write $A \overset{a}{=} B$. This amounts to equality in the quotient group PG/FG. Moreover G acts by right translation on these groups, and QG/FG is the subgroup of elements invariant under this action. Elements of QG are said to be almost invariant. We define the number of ends of G to be

$$e(G) = \dim_{\mathbb{Z}_2} (QG/FG).$$

If G is finite, all subsets are finite and clearly $e(G) = 0$. Otherwise, G is an infinite set which is invariant (not merely 'almost'), so $e(G) \geq 1$.

For finitely generated groups G we can identify these two definitions as follows. Choose a finite set S of generators, and form the Cayley graph $\Gamma_S = \Gamma(S, G)$. Clearly this is locally finite.

Proposition 5.2. $e(G) = e(\Gamma_S)$.

Corollary 5.3. $e(\mathbb{Z}) = 2$. For $\{1\}$ generates $\mathbb{Z}$, and the corresponding graph is homeomorphic to $\mathbb{R}$.

Proof. We can identify the vertices of Γ_S with elements of G, and hence $C^0(\Gamma_S)$ with PG and $C_f^0(\Gamma_S)$ with FG. What we have to show,

then, is that if the 0-cochain c corresponds to the subset A, then

δc is finite $\iff$ $A \in QG$.

Now δc is supported by the set of edges (g, gs) $(g \in G, s \in S)$ with just one end in A. For fixed s, this means that g belongs to just one of A, As^{-1}; i.e. $g \in A + As^{-1}$. If A is almost invariant, for each s we have finitely many g, hence a finite number of edges in total. Conversely if δc is finite, the class of A in PG/FG is invariant under each s^{-1} $(s \in S)$, hence under the group, G, which they generate.

This connection can now be extended.

Theorem 5.4. Let G act freely on the connected complex X, with finite quotient K (equivalently, $X \to K$ is a connected regular covering, with group G). Then $e(G) = e(X)$.

Proof. As before, we may suppose X a graph by ignoring cells of dimension > 1. Let T be a maximal tree in K, $\tilde{T}$ a lift to X. The trees $g\tilde{T}$ $(g \in G)$ are all disjoint; if we identify each to a point (obtaining Y, say) $H^0_e(X)$ is unaltered. For if c has finite coboundary, it is constant on all but finitely many $g\tilde{T}$, hence almost equal to a c' which is constant on each. The natural map $C^0Y \to C^0X$ preserves the subgroups of finite cochains and of cochains with finite coboundaries, hence induces $H^0_e(Y) \to H^0_e(X)$. This is clearly injective, and the above observation proves it surjective.

We may thus suppose K/T has only one vertex. But now Y can be identified with a suitable graph Γ_S, and the result follows from (5.2).

Corollary 5.5. If G acts freely on $\mathbb{R}^n$ with compact quotient, e.g. if $G = \mathbb{Z}^n$, we have $e(G) = 1$.

The connection with topology is valid only for finitely generated G. However, an interpretation in terms of group cohomology can always be given. For any G, $H^n(G; PG) \cong \mathbb{Z}_2$ $(n = 0)$, 0 $(n \neq 0)$. Moreover FG can be identified with the group ring $\mathbb{Z}_2G$. The invariant subgroup $QG/FG = H^0(G; PG/\mathbb{Z}_2G)$, and for G infinite, since $H^0(G; \mathbb{Z}_2G) = 0$, we deduce that

$$e(G) = 1 + \dim H^1(G; \mathbb{Z}_2 G).$$

We now begin work on calculating the number of ends of various groups. We start with lemmas noting the invariance of $e(G)$ under commensurability and isogeny.

Lemma 5.6. If H is a subgroup of finite index in G, $e(G) = e(H)$.

Proof. If G is finitely generated, we may use (5.2) and observe that H acts freely on Γ_S with finite quotient (a covering of $G\backslash\Gamma_S$, of degree $|G : H|$).

In general, if $A \subset G$ is almost invariant in G so is $A \cap H$ in H. For $(A \cap H) + (A \cap H)h = (A + Ah) \cap H$ is finite, for any $h \in H$. Thus intersection induces a homomorphism $QG/FG \xrightarrow{\phi} QH/FH$. Choose a left transversal T for H in G.

Now ϕ is injective, for if $A \cap H$ is finite so is each $Ag \cap H$, hence $A \cap Hg^{-1}$. Letting g run through the finitely many elements of T^{-1}, we deduce A is finite.

And ϕ is surjective, for if $B \subset H$ is almost invariant consider $A = BT$. Certainly $A \cap H = B$. For any $g \in G$, $t \in T$, write $tg = h_t s$ $(s \in T)$. Then $A + Ag = \Sigma(Bt + Btg) = \Sigma(Bt + Bh_t s)$. But Bh_t is almost equal to B, and s runs through T as t does. Hence $A + Ag$ is finite.

Lemma 5.7. If K is a finite normal subgroup of G,

$$e(G) = e(G/K).$$

Proof. Write $p : G \to G/K$ for the natural map and $p_t : PG \to P(G/K)$, $p^{-1} : P(G/K) \to PG$ for the direct and inverse image maps induced by p. Then $p_t p^{-1} B = B$ for any $B \subset G/K$, while for $A \subset G$, $p^{-1} p_t A = AK$. Trivially B is almost invariant $\iff p^{-1}B$ is, and if A is almost invariant it is almost equal to AK, so $p_t(A)$ is almost invariant. Hence p_t, p^{-1} preserve the subgroups Q and F and the induced maps between QG/FG and $Q(G/K)/F(G/K)$ are two sided inverses, hence isomorphisms.

We now come to the main result of this section.

Theorem 5.8. *Suppose* G *finitely generated,* $A \in QG$ *such that both* A *and* A^* $(= G - A)$ *are infinite, and that* $H = \{h \in G : hA \overset{a}{=} A\}$ *is infinite. Then* G *has an infinite cyclic subgroup of finite index.*

Since the left translations on PG commute with the right, there is an induced action of G from the left on QG/FG. As H is the stabilizer of A for this action, it is a subgroup.

Corollary 5.9. *If* G *is finitely generated,* $e(G) = 0, 1, 2$ *or* ∞.

For suppose $e(G) \neq 0$, 1 or ∞. Then G is infinite $(e(G) \neq 0)$ and acts on the finite $(e(G) \neq \infty)$ group QG/FG. As $e(G) \neq 1$, we can find an A as above; the isotropy group H has finite index in G, so is infinite. Then by (5.8) there is a subgroup Z of finite index which is infinite cyclic, by (5.6) $e(G) = e(Z)$ and by (5.3), $e(Z) = 2$.

We also have a characterization of groups with 0 ends (finite) or 2 ends (finite extensions of $\mathbb{Z}$). Our next main objective will be a study of groups with ∞ ends. The restriction to finitely generated groups is not essential: the result of Corollary 5.9 is proved in Cohen's book [5] for groups which are not locally finite; he also shows that a countable locally finite group has ∞ ends, and (see Goalby [31]) an uncountable one also has 1 or ∞.

We begin the proof of (5.8) with a lemma, which (together with the corollary) will also be repeatedly used in chapter 6.

Lemma 5.10. *Let* $A_0, A_1 \in QG$. *For almost all* $g \in A_0$, *either* $gA_1 \subseteq A_0$ *or* $gA_1^* \subseteq A_0$.

Proof. Choose a finite set S of generators of G, and use (as in (5.2)) the action of G on Γ_S. Pick *connected* finite subgraphs C_i of Γ_S containing δA_i.

For each vertex c of C_1, $gc \in A_0$ for almost all $g \in A_0$. As C_i is finite, $gC_1 \cap C_0 = \phi$ for almost all $g \in G$. Hence for almost all $g \in A_0$, we have $gC_1 \cap C_0 = \phi$, and $gc \in A_0$ for each vertex c of C_1.

For any collection A of vertices of Γ, let $\overline{A}$ denote the maximal subgraph of Γ with vertex set equal to A. Each component E of $\overline{A_1}$ or $\overline{A_1^*}$ contains a vertex of C_1, so gE meets A_0: if it also meets A_0^*,

it meets C_0. But C_0 is connected and disjoint from gC_1, so lies in a single component gE. Thus A_0^* cannot meet both gA_1 and gA_1^*.

Corollary 5.11. If $A_0, A_1 \in QG$, then for almost all $g \in G$ one (at least) of $gA_1 \subset A_0$, $gA_1^* \subset A_0$, $gA_1 \subset A_0^*$, $gA_1^* \subset A_0^*$ holds.

Proof of 5.8. Interchanging A, A^* if necessary, we may assume $H \cap A$ infinite. We may also adjoin 1 to A. By the lemma, for almost all $g \in A$ either $gA \subseteq A - \{1\}$ or $gA^* \subseteq A - \{1\}$. Hence we can choose $c \in H \cap A$ satisfying one of these: necessarily $cA \subseteq A - \{1\}$. We will show that c generates the required subgroup.

If $n > 0$, $c^n A \subseteq cA \subset A$. Thus $c^n \neq 1$, so c has infinite order. As $1 \in A$, $c^n \in A$ for $n > 0$, and as $c^n A \subseteq A - \{1\}$ for $n > 0$, we have $c^{-n} \in A^*$ for $n > 0$.

If $d \in \cap\{c^n A : n > 0\}$, then $c^{-n} \in Ad^{-1}$ for $n > 0$, contradicting the fact that $Ad^{-1} + A$ is finite, and all the c^n distinct. Hence $\cap\{c^n A : n > 0\} = \phi$. So

$$A = \cup\{c^n A - c^{n+1} A : n \geq 0\}$$
$$= \cup\{c^n(A - cA) : n \geq 0\}$$

is contained in the union of finitely many (right) cosets of $\langle c \rangle$ in G: recall that $c \in H$, so $A - cA$ is finite. The same holds for A^* (replacing c by c^{-1}). Hence the infinite cyclic subgroup $\langle c \rangle$ has finite index in G.

Alternate proof of 5.8. As before, we may assume that $H \cap A$ is infinite. Lemma 5.10 and its proof tells us that for almost all $g \in A$, $g(\delta A) \cap \delta A$ is empty and either $gA \subset A$ or $gA^* \subset A$. Hence there is an element c of $H \cap A$, such that $c(\delta A) \cap \delta A$ is empty and either $cA \subset A$ or $cA^* \subset A$. As cA is almost equal to A, we must have $cA \subset A$ and the inclusion must be strict, as $c(\delta A) \cap \delta A = \phi$. Let $B = A + cA$. Then B is non-empty, finite and $B \subset A$, $B \cap cA = \phi$. Further for any two integers r, s, with $r > s$, we have $c^r B \cap c^s B = \phi$. For $c^r B \cap c^s B = c^s(c^{r-s} B \cap B)$, and $c^{r-s} B \subset c^{r-s} A \subset cA$. Thus $c^{r-s} B \cap B \subset cA \cap B = \phi$.

Now consider $\sum_{n\in\mathbb{Z}} c^n B$, which equals $\bigcup_{n\in\mathbb{Z}} c^n B$ by the above. As δ is additive, we have $\delta\left(\sum_{n\in\mathbb{Z}} c^n B\right) = \sum_{n\in\mathbb{Z}} \left(c^n \delta A + c^{n+1}\delta A\right) = 0$. (Note that these infinite sums make sense.) Thus $\bigcup_{n\in\mathbb{Z}} c^n B$ must equal G, and so the cyclic subgroup of G generated by c has index equal to the order of B. As G is infinite, c must have infinite order and the result follows.

One can also give a more direct proof of the result (5.9) that a finitely generated group must have 0, 1, 2 or ∞ ends. Let G be a finitely generated, infinite group and suppose that $e(G)$ is a positive integer n. Choose a finite generating set for G and let Γ be the corresponding graph. G acts on Γ on the left with finite quotient. Let L be a finite connected subgraph of Γ such that $\Gamma - L$ consists of n infinite components $V_1, \ldots, V_n$. As G is infinite, there exists $g \in G$ with $gL \cap L = \phi$. Thus gL lies in one of the V's, V_1 say. Exactly one of the components of $V_1 - gL$ is infinite, for $\Gamma - (L \cup gL)$ has only n infinite components. Now $L \cup V_2 \cup \ldots \cup V_n$ is connected, so that $\Gamma - gL$ has at most two infinite components. As g is a homeomorphism, $\Gamma - L$ must have at most two infinite components and this proves the required result.

We finish this section by giving some more information about groups with two ends.

Theorem 5.12. The following conditions on a finitely generated group G are equivalent:

(i) $e(G) = 2$,

(ii) G has an infinite cyclic subgroup of finite index,

(iii) G has a finite normal subgroup with quotient $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$,

(iv) $G = F *_F$ with F finite, or $G = A *_F B$ with F finite and $|A : F| = |B : F| = 2$.

Proof. (i) $\Rightarrow$ (ii) by Theorem 5.8, for H will have index at most 2 in G.

(ii) $\Rightarrow$ (i) by Lemma 5.6 and the fact that $e(\mathbb{Z}) = 2$.

(iii) $\Rightarrow$ (iv) If F is a finite normal subgroup of G with quotient

$\mathbb{Z}$, then $G = F *_F$. If the quotient is $\mathbb{Z}_2 * \mathbb{Z}_2$, then $G = A *_F B$, where A and B are the inverse images of the $\mathbb{Z}_2$-factors. Thus $|A : F| = |B : F| = 2$ as required.

(iv) $\Rightarrow$ (iii) If $G = F *_F$ with F finite, then both inclusions of F in F must be isomorphisms and so F is normal in G with quotient $\mathbb{Z}$. If $G = A *_F B$, with F finite and $|A : F| = |B : F| = 2$, then F is normal in A and B. Hence F is normal in G and $G/F \cong (A/F) * (B/F) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

(iii) $\Rightarrow$ (i) by Lemma 5.7. Note that $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $D(\infty)$, the infinite dihedral group, and so has two ends.

Finally, we prove (ii) $\Rightarrow$ (iii), to complete the theorem.

First, G must contain an infinite cyclic subgroup K of finite index which is also normal in G. One takes for K the intersection of all the conjugates of the original infinite cyclic subgroup. Let H denote the centralizer of K in G. Thus $|G : H| \leq 2$. H is finitely generated and its centre is a subgroup of finite index. A theorem of Schur (see e. g. W. R. Scott, Group Theory, Prentice-Hall, 1964, §15.1.13) tells us that H', the commutator subgroup of G, is finite. Now H/H' must have rank 1, and so there is an epimorphism $\phi : H \to \mathbb{Z}$ with finite kernel L. If $G = H$, our result is proved. Otherwise observe that H is normal in G and L is characteristic in H, as L is the torsion subgroup of H. Thus L is normal in G and we have the exact sequence

$$1 \to H/L \to G/L \to \mathbb{Z}_2 \to 1.$$

We know that G/L must be non-abelian, and therefore G/L is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. This completes the proof of Theorem 5.12.

6. THE STRUCTURE THEOREM FOR GROUPS WITH INFINITELY MANY ENDS

The aim of this section is to describe which finitely generated groups have infinitely many ends. The neatest formulation of the result includes the case of two ends.

Theorem 6.1. If G is a finitely generated group, then $e(G) \geq 2$ if and only if G splits over a finite subgroup.

Remark. Theorem 5.12 tells us that $e(G) = 2$ if and only if either $G = F *_F$ with F finite or $G = A *_F B$ with F finite and $|A : F| = |B : F| = 2$.

This remarkable result is due to Stallings [37], [26], but our treatment of the proof is an amalgam of results of Cohen [5], Dunwoody [7] and Stallings [26].

There is a close connection between Theorem 6.1 and the Sphere Theorem. In fact, Stallings discovered the result by considering the proof of the Sphere Theorem due to Papakyriakopoulos [19] and Whitehead [29].

Sphere Theorem. If M is an orientable 3-manifold with $\pi_2(M) \neq 0$, there is an embedded 2-sphere S in M which represents a non trivial element of $\pi_2(M)$.

Let M be a closed orientable 3 manifold with fundamental group G. One can show easily (see below) that the hypothesis that $\pi_2(M)$ is non-zero is equivalent to asserting that $e(G) \geq 2$. Also the conclusion of the Sphere Theorem implies that G splits over the trivial subgroup. Thus the Sphere Theorem is extremely like Theorem 6.1, when M is a closed manifold. Further, it is possible [26] to give a proof of the Sphere Theorem which uses Theorem 6.1.

The reason why $\pi_2(M) \neq 0$ if and only if $e(G) \geq 2$ is as follows. Let $\tilde{M}$ denote the universal covering space of M. Then Theorem 5.4 tells us that $e(G) = e(\tilde{M})$. For these purposes it will be convenient to use coefficients $\mathbb{Z}$, not $\mathbb{Z}_2$, when defining the groups $H^n(\tilde{M})$, $H_f^n(\tilde{M})$, $H_e^n(\tilde{M})$. The natural analogue of Proposition 5.1 is that $e(\tilde{M})$ equals the rank of $H_e^0(\tilde{M})$, where the rank of an abelian group is defined to be the maximal rank of all finitely generated free abelian subgroups or ∞ if this maximum does not exist. Now consider the long exact sequence connecting the groups $H^n(\tilde{M})$, $H_f^n(\tilde{M})$, $H_e^n(\tilde{M})$. This begins

$$H_f^0(\tilde{M}) \to H^0(\tilde{M}) \to H_e^0(\tilde{M}) \to H_f^1(\tilde{M}) \to H^1(\tilde{M}) \to \dots .$$

As $H^1(\tilde{M}) = 0$, we see that $e(\tilde{M}) \geq 2$ if and only if $H_f^1(\tilde{M})$ is non-zero. Now Poincaré duality for $\tilde{M}$ gives an isomorphism between $H_f^1(\tilde{M})$ and $H_2(\tilde{M})$, and we have $H_2(\tilde{M}) \cong \pi_2(\tilde{M}) \cong \pi_2(M)$. Thus $\pi_2(M) \neq 0$ if and only if $e(G) \geq 2$.

The conclusion of the Sphere Theorem implies that $G \cong A *_{\pi_1(S)} B$ or $A *_{\pi_1(S)}$ according to whether S separates M or not. As $\pi_1(S)$ is trivial, this implies that G splits over the trivial subgroup unless S separates M into two components one of which is simply connected. We show that this is impossible. If this did happen, we would have a compact, simply connected 3-manifold X with boundary a 2-sphere. Hence Poincaré duality tells us that $H_2(X, \partial X) \cong H^1(X) = 0$. Now the exact homology sequence of the pair $(X, \partial X)$

$$0 \to H_3(X, \partial X) \to H_2(\partial X) \to H_2(X) \to H_2(X, \partial X) \to \ldots$$

shows that $H_2(X) = 0$. The Hurewicz Theorem then implies that $\pi_2(X) = 0$, so that S is null-homotopic in X, contradicting our assumption on S.

A purely group theoretic result, which follows easily from Theorem 6.1 is the following.

Theorem 6.2. If G is a finitely generated, torsion free group with a free subgroup of finite index, then G is free.

Remark. Swan [27] has extended this result by removing the restriction that G be finitely generated.

Proof. Let $\mu(G)$ denote the minimal number of generators of G. If $\mu(G) = 0$, then G is trivial, so the theorem holds. If $\mu(G) > 0$, then G is non-trivial. As G cannot be finite, G has a non-trivial free subgroup F of finite index and so $e(G) \geq 2$. Thus Theorem 6.1 tells us that G splits over a finite subgroup. Now the only finite subgroup of G is the trivial subgroup, so that either $G \cong \mathbb{Z}$ or $G \cong G_1 * G_2$, where each G_i is non-trivial. In the first case, our result is proved. In the second case, we observe $\mu(G_i) < \mu(G)$ by Corollary 2.1 and that $F \cap G_i$ is a free subgroup of G_i which is of finite index. Thus the required result follows by induction on $\mu(G)$. Note that we have used the fact that

subgroups of free groups are free.

We now come to the proof of Theorem 6.1 and we start with the easy half.

Lemma 6.3. If G splits over a finite subgroup, then $e(G) \geq 2$.

Remark. For this result, G need not be finitely generated.

Proof. It suffices to produce an almost invariant subset E of G such that E and E^* are infinite.

First suppose that $G = A *_C B$, where C is finite, and recall the canonical form for elements of G given by Theorem 1.6. One chooses based transversals T_A and T_B for C in A and B and obtains the form $a_1 b_1 \ldots a_n b_n c$ for any element of G, where $c \in C$, $a_i \in T_A$, $b_i \in T_B$ and $a_i = 1 \Rightarrow i = 1$, $b_i = 1 \Rightarrow i = n$. Let E be the subset of G consisting of elements for which a_1 is non-trivial. Clearly E and E^* are infinite. (This uses the fact that $A \neq C \neq B$.) If $b \in B$, then $Eb = E$. If $a \in A$, then $Ea \subset E \cup C$ and so also $Ea^{-1} \subset E \cup C$. Hence

$$E \subset Ea \cup Ca \subset E \cup C \cup Ca$$

so that $E \overset{a}{=} Ea$. As A and B together generate G, we have $Eg \overset{a}{=} E$ for all g in G.

Secondly suppose that $G = A *_C$, where C is finite, and recall the canonical form for elements of G given by Theorem 1.7. One chooses based transversals T_i of $\alpha_i(C)$ in A and obtains the form $a_1 t^{\varepsilon_1} a_2 t^{\varepsilon_2} \ldots a_n t^{\varepsilon_n} a_{n+1}$, where $a_{n+1} \in A$, $a_i \in T_1$ if $\varepsilon_i = 1$, $a_i \in T_2$ if $\varepsilon_i = -1$, and moreover $a_i \neq 1$ if $\varepsilon_{i-1} \neq \varepsilon_i$. Let E be the subset of elements of G for which a_1 is trivial and $\varepsilon_1 = 1$. If $a \in A$, then $Ea = E$. Also $Et \subset E$ and $Et^{-1} \subset E \cup \alpha_1(C)$. Therefore, as before, E is almost invariant in G. This completes the proof of Lemma 6.3.

The hard part of Theorem 6.1 is the result that a finitely generated group G with infinitely many ends must split over a finite subgroup. Our aim, following Dunwoody [7], is to produce a tree T on which G acts with quotient a single edge. We start by considering graphs and trees in more detail than before. Let Γ be an abstract graph.

Definition. An edge path in Γ is a sequence $e_1, \ldots, e_n$ of edges such that $\partial_1 e_i = \partial_0 e_{i+1}$ and $e_i \neq \bar{e}_{i+1}$, for $i = 1, 2, \ldots, n-1$. If e, f are edges of Γ, we will write $e \leq f$ if there is an edge path with $e_1 = e$ and $e_n = f$.

The relation $\leq$ has the following properties.

(A) For any graph Γ, the relation $\leq$ is reflexive and transitive.

(B) For any graph Γ and any edges e, f of Γ, if $e \leq f$ then $\bar{f} \leq \bar{e}$.

(C) The graph Γ is connected if and only if for any pair e, f of edges of Γ, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds.

(D) The graph Γ has no circuits if and only if whenever $e \leq f$ and $f \leq e$, then $e = f$.

(E) If Γ has no circuits, then for no pair e, f of edges can we have $e \leq f$ and $e \leq \bar{f}$.

(F) If Γ has no circuits, then for any pair e, f of edges there are only finitely many edges g with $e \leq g \leq f$.

If a relation satisfies the conditions in (A) and (D), we shall call it a partial order. Thus if Γ is a tree, the relation $\leq$ on $E(\Gamma)$ is a partial order and the following conditions hold.

(1) If $e \leq f$, then $\bar{f} \leq \bar{e}$.

(2) If $e, f \in E(\Gamma)$, there are only finitely many $g \in E(\Gamma)$ such that $e \leq g \leq f$.

(3) If $e, f \in E(\Gamma)$, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds.

(4) If $e, f \in E(\Gamma)$, we cannot have $e \leq f$ and $e \leq \bar{f}$.

Remark. If two of the inequalities in (3) hold, then $e = f$ or $e = \bar{f}$.

The next step is to show that if we start with a partially ordered set E satisfying all the above conditions, then we can construct a tree out of E.

Let E be a partially ordered set with an involution $e \to \bar{e}$, where $e \neq \bar{e}$, and suppose that conditions (1)-(4) hold. Write $e < f$ if $e \leq f$ and $e \neq f$. Write $e << f$ if $e < f$, and $e \leq g \leq f$ implies $g = e$ or $g = f$. We need the following technical result.

Lemma 6.4. The relation $\sim$ on E, defined by $e \sim f$ if and only if $e = f$ or $e << \bar{f}$, is an equivalence relation.

Proof. The relation is obviously reflexive and is symmetric because $e << \bar{f}$ implies $f << \bar{e}$.

We suppose $e \sim f$ and $e \sim h$ and will show $f \sim h$. The first step is to show that $f = h$ or $f < \bar{h}$. Condition (3) implies that one of $f \leq h$, $\bar{f} \leq \bar{h}$, $\bar{f} \leq h$, $f < \bar{h}$ holds. If $f \leq h$, then $\bar{h} \leq \bar{f}$ and we would have $e < \bar{h} \leq \bar{f}$ which implies $h = f$ as $e << \bar{f}$. If $\bar{f} \leq \bar{h}$, then $e < \bar{f} \leq \bar{h}$ implies $h = f$ as $e << \bar{h}$. If $\bar{f} \leq h$, we would have $e < \bar{f} \leq h$. As $e \leq \bar{h}$, this contradicts (4). The only remaining possibility is $f < \bar{h}$. Hence either $f = h$ or $f < \bar{h}$ as required.

The last step is to suppose $f < \bar{h}$ and $f \leq g \leq \bar{h}$ and prove $g = f$ or $g = \bar{h}$. One of the inequalities $\bar{e} \leq g$, $\bar{e} \leq \bar{g}$, $e < g$, $e < \bar{g}$ must hold. If $\bar{e} \leq g$, then $\bar{e} \leq \bar{h}$. As $e \leq \bar{h}$, this contradicts (4). If $\bar{e} \leq \bar{g}$, then $\bar{e} \leq \bar{f}$ which again contradicts (4). If $e < g$, then $e < g \leq \bar{h}$ shows that $g = \bar{h}$ as $e << \bar{h}$. If $e < \bar{g}$, then $e < \bar{g} \leq \bar{f}$ shows that $g = f$ as $e << \bar{f}$. Hence $g = f$ or $g = \bar{h}$ as required. This completes the proof of Lemma 6.4.

We construct a graph Γ out of $\mathbf{E}$ as follows. Let $t(e)$ denote the equivalence class of e in $\mathbf{E}$ under the relation $\sim$. Let $V = \{t(e): e \in \mathbf{E}\}$ and let $\partial_0 e = t(\bar{e})$. Then the sets $\mathbf{E}$, V and the map ∂_0 form an abstract graph Γ.

Theorem 6.5. Γ is a tree and the order relation which Γ induces on $\mathbf{E}$ is the same as the original relation.

Proof. We will prove the second part of the theorem. The fact that Γ is a tree will then follow from properties (C) and (D). Lemma 6.4 tells us that for distinct elements e, f of $\mathbf{E}$ we have $t(f) = t(e)$ if and only if $e << \bar{f}$. Thus $\partial_1 e = \partial_0 f$ if and only if $e << f$. It follows that if e and f are joined by an edge path in Γ, then $e \leq f$, and condition (2) shows that if $e \leq f$, then e and f can be joined by an edge path in Γ. Hence Γ does induce the original order relation on $\mathbf{E}$, as required.

Consider the following examples of Theorem 6.5 in action.

Let G be the free group of rank 2 with generators a, b and let Γ be the corresponding graph for G. If e is an edge of Γ it separates Γ into two components. Thus $G = A \cup A^*$ where $\delta A = \delta A^* = e$.

Let E be the set of all subsets A of G with δA equal to a single edge of Γ. We partially order E by inclusion. Then E has an involution $A \to A^*$ and satisfies conditions (1)-(4). The tree constructed in Theorem 6.5 is the graph Γ again.

One can obtain an action of G on a different tree T as follows. Let A be the subset of G which contains a and whose coboundary is the edge (e, a) of Γ. Let F be the set of all translates gA, gA^* of A and A^*, partially ordered by inclusion. Again F satisfies conditions (1)-(4). Hence we can construct a tree T from Theorem 6.5. The natural left action of G on F induces an action of G on T which has quotient a single edge. Let g be an element of G such that $gA = A$ or A^*. Then $g(\delta A) = \delta A$ so that the edges (g, ga) and (e, a) are equal. This is only possible if $g = e$ so that G acts on T without inversions and the stabilizer of any edge of T is trivial.

Recall that $t(A)$ consists of A together with every element B of F such that $A << B^*$. We will call an edge of Γ of the form (g, ga) an a-edge, and an edge of the form (g, gb) a b-edge. If $A << B^*$, then δB is an a-edge of Γ which has no vertices in A and such that the path from δB to δA consists only of b-edges. It is now easy to see that

$$t(A) = \{b^n A : n \in \mathbb{Z}\} \cup \{b^n a^{-1} A^* : n \in \mathbb{Z}\}.$$

Hence the stabilizer of $t(A)$ is the infinite cyclic subgroup H of G generated by b. One can also see that $\partial_0 A = \partial_1 A^* = a\partial_1 A$. Hence G acts transitively on the vertices of T, so that $G\backslash T$ is a loop. This action of G on T corresponds to expressing G as $H *_{\{1\}}$. The graph T is obtained from Γ by identifying each b-edge of I to a point.

We can now sketch the proof of the main part of Theorem 6.1, i.e. if G is a finitely generated group and $e(G) = \infty$, then G splits over a finite subgroup. The idea is to proceed, as in the example above, to construct a tree T on which G acts so that the stabilizer of any edge is finite and the quotient $G\backslash T$ is a single edge. However, we need to partially order our almost invariant sets by almost inclusion and not by strict inclusion in order to be able to prove that condition (3) holds.

For any almost invariant set B of G, write $[B]$ for the set of all almost invariant sets of G which are almost equal to B. Define $[B] \leq [C]$

if and only if $B \overset{a}{\subset} C$. Fix a proper almost invariant set A of G i.e. A and A^* are infinite, and let E be the set of $[gA]$ and $[gA^*]$ for all g in G, partially ordered by $\leq$. We have the involution $[A] \to [A^*]$ on E. Our aim is to choose A so that E satisfies conditions (1)-(4). Assuming that we can do this, we can construct the required tree T. The stabilizer of the edge $[A]$ will be finite because $\{g \in G : gA \overset{a}{=} A\}$ is finite by Theorem 5.8. G may invert edges but if so we simply subdivide T. Hence, by Theorem 4.3, G is an amalgamated free product over a finite subgroup F. Hence G splits over F so long as no vertex of T has stabilizer equal to G. We show that this is impossible.

Suppose that G fixes a vertex v of T. Then every edge of T has v as a vertex. In particular, v must be one of the original vertices of T, and was not introduced by subdivision. Now we consider the original T. If e and f are edges of T with $e < f$, we have $e << f$. Now Lemm 5.10 tells us that for almost all elements x of A, $xA \subset A$ or $xA^* \subset A$. As $\{g \in G : gA \overset{a}{=} A\}$ is finite, by Theorem 5.8, we deduce that there is an element x of A such that xA is not almost equal to A or A^* and either $xA \subset A$ or $xA^* \subset A$. Similarly, there is an element y of A^* such that yA^* is not almost equal to A or A^* and either $yA \subset A^*$ or $yA^* \subset A^*$. If $xA \subset A$, then we have $x^2A \subset xA \subset A$ and this contradicts the fact that if e, f are edges of T with $e < f$, then $e << f$. Similarly, if $yA^* \subset A^*$, we obtain a contradiction. If $xA^* \subset A$ and $yA \subset A^*$, then $xyA \subset xA^* \subset A$ and again we have a contradiction. Therefore G cannot fix a vertex of T.

In order to complete the proof of Theorem 6.1, we must show how to find a proper almost invariant set A in G such that the partially ordered set E satisfies conditions (1)-(4). Conditions (1) and (4) hold automatically for any choice of A. Our next result says that condition (2) also holds for any choice of A.

Lemma 6.6 *Let G be a finitely generated group with infinitely many ends. If B, C, D are proper almost invariant subsets of G, then $\{g \in G : B \overset{a}{\subset} gC \overset{a}{\subset} D\}$ is finite.*

Proof. The result is trivial if B is not almost contained in D. If

B is strictly contained in D, we can add an element of D^* to B without altering the problem. Hence we suppose that $B \overset{a}{\subset} D$ but $B \not\subset D$.

If $B \overset{a}{\subset} gC \overset{a}{\subset} D$, then either $gC \not\subset D$ or $B \not\subset gC$. We will show that $\{g \in G : gC \overset{a}{\subset} D,\ gC \not\subset D\}$ and $\{g \in G : B \overset{a}{\subset} gC,\ B \not\subset gC\}$ are both finite. This will prove the required result. Now Corollary 5.11 states that for almost all elements g in G one of $gC \subset D$, $gC \subset D^*$, $gC^* \subset D$, $gC^* \subset D^*$ holds. If $gC \overset{a}{\subset} D$ and $gC \not\subset D$, none of these four inclusions can hold except for $gC^* \subset D^*$. If $gC \overset{a}{\subset} D$ and $gC^* \subset D^*$, then $gC \overset{a}{=} D$. As $\{g \in G : gC \overset{a}{=} C\}$ is finite, by Theorem 5.8, we deduce that $\{g \in G : gC \overset{a}{\subset} D,\ gC \not\subset D\}$ is finite. Similarly $\{g \in G : B \overset{a}{\subset} gC,\ B \not\subset gC\}$ is finite.

Finally, we show that it is possible to choose A so that E satisfies condition (3). Note that for almost all g in G, we know that one of $gA \subset A$, $gA \subset A^*$, $gA^* \subset A$, $gA \subset A^*$ holds. We must arrange that this holds for every element of G, when we replace strict inclusion by almost inclusion.

We fix a finite generating set S for G and let $\Gamma = \Gamma(S, G)$ be the corresponding graph. If A is an almost invariant set in G, we denote the number of edges in δA by $|\delta A|$. Let k be the smallest value taken by $|\delta A|$ as A ranges over proper almost invariant sets in G. We say that a set A in G is narrow if $|\delta A| = k$.

Lemma 6.7. *Let $A_1 \supset A_2 \supset \ldots$ be a sequence of narrow sets in G. If $B = \bigcap_{n \geq 1} A_n$ is non-empty, then the sequence stabilizes, i.e. there is an integer K such that $A_n = B$, when $n \geq K$.*

Proof. Let e be an edge of δB. Then e has one vertex in every A_n and the other vertex is outside every A_n for which n exceeds some integer N. Therefore e is an edge of δA_n, when $n > N$. If δB contains $k + 1$ edges, the above argument shows that δA_n would also contain $k + 1$ edges for a suitably large value of n. It follows that $|\delta B| \leq k$ and that $\delta B \subset \delta A_n$ for all suitably large n. In particular, B is almost invariant in G. We have the equations $A_n = (A_n + B) + B$ and $\delta A_n = \delta(A_n + B) + \delta B$. As $\delta B \subset \delta A_n$ we see that $\delta(A_n + B) \cap \delta B$ is empty. As A_n is infinite, one of B and $(A_n + B)$ must be infinite.

The infinite one, X, must have $|\delta X| = k$, as X is a proper almost invariant subset of G. The other one must then have empty coboundary and so be empty. As B is non-empty, we deduce $B = A_n$, which completes the proof of the lemma.

Let g be any element of G, and let A be a narrow set in G. Then A^* is also narrow so that g must lie in a narrow set in G. Lemma 6.7 tells us that the set of all narrow subsets of G which contain g has minimal elements, where we partially order narrow sets by inclusion.

Lemma 6.8. Let A be a narrow set, minimal with respect to containing some element g of G. Then for any narrow set A_1, one of $A \overset{a}{\subset} A_1$, $A \overset{a}{\subset} A_1^*$, $A^* \overset{a}{\subset} A_1$, $A^* \overset{a}{\subset} A_1^*$ holds.

Proof. The required result is equivalent to proving that one of the sets $A \cap A_1$, $A \cap A_1^*$, $A^* \cap A_1$, $A^* \cap A_1^*$ is finite. For convenience we call these sets X_1, X_2, X_3, X_4. For each i, $\delta X_i \subset \delta A \cup \delta A_1$. As the X_i's are disjoint, any edge in $\delta A \cup \delta A_1$ has its ends in exactly two of the X_i's. Hence each edge in $\delta A \cup \delta A_1$ lies in the coboundary of exactly two of the X_i's. Hence

$$|\delta X_1| + |\delta X_2| + |\delta X_3| + |\delta X_4| = 2|\delta A \cup \delta A_1| \leq 4k,$$

where $|\delta A| = |\delta A_1| = k$.

If each X_i is infinite, then we must have $|\delta X_i| \geq k$ for each i, because each X_i^* is infinite. Hence $|\delta X_i| = k$ for each i. But one of $A \cap A_1$, $A \cap A_1^*$ (say $A \cap A_1$) is then a narrow subset of G which contains g. Hence $A \cap A_1 = A$ by the minimality of A, and so $A \cap A_1^*$ is empty - a contradiction. Therefore some X_i must be finite which completes the proof of Lemma 6.8.

In order to carry out the proof of Theorem 6.1 as sketched after Theorem 6.5, we simply need to choose a narrow set A in G which is minimal with respect to containing some element of G.

7. APPLICATIONS AND EXAMPLES

Many of the most important applications of Stallings' structure theorem for groups with infinitely many ends have to do with the cohomology of groups. We will only consider more simple minded examples. We start by discussing the problem of accessibility first posed by Wall [28].

Think of a splitting of a group over a finite subgroup as a kind of factorization. Stallings' theorem tells us that if G is finitely generated and $e(G) \geq 2$, then G has such a factorization. The first natural question to ask is whether one can go on factorizing G for ever, or whether the process of factorization must stop.

We will say that a f. g. group with at most one end is 0-accessible, and that a group G is n-accessible if G splits over a finite subgroup with each of the factor groups (n-1)-accessible. We will call a group accessible if it is n-accessible for some n.

Conjecture. Any finitely generated group is accessible.

Bamford and Dunwoody [1] have shown that accessibility is equivalent to a certain condition on the cohomology of the group, but, in general, one has no proof that their condition is satisfied. However, it is easy to see that any f. g. torsion free group G is accessible. Corollary 2. 2, which follows from Gruško's Theorem, tells us that G is a free product of indecomposables. Each factor in this decomposition has at most one end or is infinite cyclic and so G is accessible.

The following result seems to clarify the concept of accessibility.

Lemma 7. 1. *Let G be a finitely generated group. Then G is accessible if and only if G is the fundamental group of a finite graph Γ of groups, where each edge group is finite and each vertex group has at most one end.*

Proof. If Γ exists, G is obviously accessible. We prove the converse by induction on n, where G is n-accessible. If $n = 0$, we can take Γ to be a single vertex.

If $G = G_1 *_C G_2$ where G_1 and G_2 are already the fundamental

groups of graphs Γ_1 and Γ_2 of groups, and C is finite, we construct Γ as follows. By Corollary 3.8, there are vertex groups H_1, H_2 of Γ_1, Γ_2 and elements g_1, g_2 of G_1, G_2 such that $C \subset g_i^{-1} H_i g_i$, for $i = 1, 2$. By replacing every vertex and edge group H of Γ_1 by $g_1 H g_1^{-1}$ we can suppose that $C \subset H_1$, and similarly for Γ_2. Now Γ consists of Γ_1, Γ_2 and an edge e joining the vertices underlying H_1 and H_2, where e has associated group C.

If $G = A *_C$ where A is already the fundamental group of a graph Γ_1 of groups, and C is finite, we proceed as follows. We have two inclusions α_1, α_2 of C in A and each $\alpha_i(C)$ must lie in a conjugate of some vertex group H_i of Γ_1. As above, we can suppose that $\alpha_1(C) \subset H_1$, and $\alpha_2(C) \subset s H_2 s^{-1}$. (Note that possibly $H_1 = H_2$.) Now G has presentation $\{A, t : t^{-1}\alpha_1(c)t = \alpha_2(c), \forall c \in C\}$. Write $u = ts$. Then G also has presentation $\{A, u : u^{-1}\alpha_1(c)u = s^{-1}\alpha_2(c)s, \forall c \in C\}$. We replace α_2 by β_2 where $\beta_2(c) = s^{-1}\alpha_2(c)s$. As $\beta_2(C) \subset H_2$, we can take Γ to be Γ_1 together with an edge e joining the vertices of Γ_1 which underlie H_1 and H_2, where e has associated group C.

We can now re-define accessibility to allow for infinite factorization. A group is accessible if and only if it is the fundamental group of a graph of groups in which every edge group is finite and every vertex group has at most one end. For f.g. groups, this is equivalent to the old definition. One can ask if all groups are accessible, but the Kurošˇ example in Section 3 shows that the answer is negative. For Kurošˇ's group is an infinite amalgamated free product of free groups and hence is torsion free. Thus his group is accessible if and only if it can be expressed as a free product of indecomposable subgroups.

There is one other class of groups known to be accessible. That is groups with a free subgroup of finite index. We have already shown (Theorem 6.2) that if such a group is torsion free it must be free. We now state a general structure theorem for such groups. This was proved by Karrass, Pietrowski and Solitar in the f.g. case [16], Cohen in the countable case [6], and Cohen [6] and Scott [20] in the general case. See also Dunwoody [7] for a more recent proof.

Theorem 7.3. A group G has a free subgroup of finite index if and only if G is the fundamental group of a graph Γ of groups in which every vertex group of Γ is finite and the orders of all the vertex groups are bounded.

Remark. We will prove this theorem only in the case when G is f.g. We can then assume that Γ is finite, so that the boundedness condition is redundant.

Proof. Suppose that G is f.g. and has a free subgroup of finite index. We will show that Γ exists by induction on $r(G)$, where $r(G)$ is the minimal rank of free subgroups of G of finite index. If $r(G) = 0$, then G is finite and the result follows. If $r(G) > 0$, then $e(G) \geq 2$ so G splits over a finite subgroup. If $G = A *_C B$, or $G = A *_C$, I claim that $r(A)$ and $r(B)$ are each less than $r(G)$, so that the result will follow by induction as in the proof of Lemma 7.1. Let F be a free subgroup of G of finite index and of minimal rank. As C is finite, F meets any conjugate of C trivially. Hence the Subgroup Theorem applied to $F \subset A *_C B$ or $A *_C$ tells us that $F = (F \cap A) * (F \cap B) * K$ or $F = (F \cap A) * K$, for some subgroup K of F. Hence the ranks of $F \cap A$ and $F \cap B$ are each less than that of F unless one of them equals F. But then we would have F contained in A or B which is impossible as F has finite index in G, but A and B have infinite index in G.

Now suppose that G is the fundamental group of a finite graph Γ of finite groups. We use induction on the number n of edges of Γ. If $n = 0$, then G is finite and the result is obvious.

If $n \geq 1$, we pick an edge e of Γ with associated group C. Then $G = A *_C B$ or $A *_C$, according to whether e separates Γ or not, where A and B are the fundamental groups of the subgraphs of Γ obtained by removing e. Thus, by our induction hypothesis, each of A and B has a free subgroup of finite index and hence a normal free subgroup of finite index. Let A_1, B_1 denote the quotients of A and B by their normal free subgroups of finite index. As C is finite, the natural maps from A and B to A_1 and B_1 both inject C. Hence we have a natural map $A *_C B \to A_1 *_C B_1$ or $A *_C \to A_1 *_C$, which injects any finite subgroup

of G. Lemma 7.4 below tells us that there are maps of $A_1 *_C B_1$ or $A_1 *_C$ to a finite group which inject A_1 and B_1. By composing these maps we obtain a homomorphism from G to a finite group which injects every finite subgroup of G. The kernel of this homomorphism must be a free group, by the Subgroup Theorem, which completes the proof of Theorem 7.3.

Lemma 7.4. *If $G = A *_C B$ or $A *_C$, with A, B and C finite, then G has a free subgroup of finite index.*

Proof. We construct a homomorphism from G to a finite group which injects A and B. The kernel must be free, by the Subgroup Theorem.

Case $G = A *_C B$

Let $X = A/C \times C \times B/C$, where A/C denotes the set of all cosets aC of C in A. We will represent A and B faithfully as permutation groups of the finite set X, in such a way that C acts on X in the same way for each action. There will then be a homomorphism $G \to S(X)$, the group of permutations of X, which injects A and B.

Choose a transversal $t : A/C \to A$. We have a bijection $A/C \times C \to A$ sending (α, c) to $t(\alpha)c$. The action of A on itself by right multiplication gives an action of A on $A/C \times C$, by using this bijection. We let A act on X by defining $(\alpha, c, \beta)a = ((\alpha, c)a, \beta)$. If $c' \in C$, then $(\alpha, c, \beta)c' = (\alpha, cc', \beta)$. Similarly we use a transversal of C in B to define an action of B on X. For this action also, we have $(\alpha, c, \beta)c' = (\alpha, cc', \beta)$ for all $c' \in C$.

Case $G = A *_C$

We have two injections of C into A. We use one of them to identify C with a subgroup of A. Thus we have a subgroup C of A and an injective map $\phi : C \to A$, whose image we denote by C_1.

Let $X = A$, and let A act on X by right multiplication. The two induced actions of C are each multiples of the right regular representation so are equivalent. We can write down an equivalence as follows. Choose transversals $T : A/C \to A$, $T_1 : A/C_1 \to A$ and a bijection $\psi : A/C \to A/C_1$

Then T, T_1 induce bijections $U : A/C \times C \to A$, $U_1 : A/C_1 \times C_1 \to A$ (as above), and we define θ to be the composite

$$A \xrightarrow{U^{-1}} A/C \times C \xrightarrow{\psi \times \phi} A/C_1 \times C_1 \xrightarrow{U_1} A.$$

Then $\theta(ac) = \theta(a)\phi(c)$. Now we can define a homomorphism $r : G \to S(X)$ by letting $r|A$ be the right regular representation and $r(t) = \theta$.

Remark. These constructions - which do not depend on finiteness (except to suppose the existence of a bijection ψ) - give an alternative proof of the assertion (1.6, 1.7) that if $C \to A$, $C \to B$ are injective, so are $A \to A *_C B$, $A \to A *_C$.

Having discussed the accessibility of groups i.e. the existence of a factorization, the next question to consider is that of uniqueness of the factorization. One would like some analogue of Theorem 3.5 for free products. The first point is that given any graph Γ of groups with fundamental group G, one can construct a larger graph Γ', also with fundamental group G by adding an edge e to Γ with only one vertex of e in Γ and an isomorphism at the other end of e. This corresponds to expressing G as $G *_C C$ for some subgroup C.

We will say that an edge e in a graph of groups Γ is <u>trivial</u> if the two ends of e are distinct vertices of Γ and e has an isomorphism at one end. If Γ has such an edge, we can replace Γ by a new graph Γ' obtained from Γ by identifying e to a point, such that Γ' has the same fundamental group as Γ. Hence if we start with a finite graph Γ, we can eliminate all the trivial edges. However, this is false for infinite graphs. For example, let Γ be the graph with vertices $1, 2, \ldots$ and edges e_i joining i to $i+1$. We associate an infinite cyclic group A_i to the vertex i and an infinite cyclic group B_i to the edge e_i. The map $B_i \to A_i$ is an isomorphism and the map $B_i \to A_{i+1}$ is multiplication by two. The fundamental group of Γ' is the dyadic rationals, but every edge of Γ is trivial.

We will say that a graph of groups with no trivial edges is <u>minimal.</u> Then any finitely generated accessible group G is the fundamental group of some minimal graph Γ, where each edge group is finite and each vertex group has at most one end. Even with minimal graphs, one still

cannot expect that the graph Γ is unique. For example, if $G = G_1 * \ldots * G_n$, then one can take for Γ any tree with n vertices, and associate $G_1, \ldots, G_n$ to the vertices and the trivial group to each edge. The same problem arises for amalgamated free products e. g. when $G = G_1 *_C \cdots *_C G_n$. Also if $G = A *_C B *_D E$ with $D \subset C$, then $G = E *_D A *_C B$ giving two possible graphs for G.

We need the following result.

Lemma 7. 5. *Let G be the fundamental group of a finite graph Γ of groups, such that each edge group is finite and each vertex group has at most one end.*

(i) *If A is a subgroup of G with at most one end, then A lies in a conjugate of a vertex group of Γ.*

(ii) *Let v_1, v_2 be vertices of Γ with associated groups G_1, G_2. If $A = G_1 \cap G_2^g$, then either there is an edge path in Γ from v_1 to v_2 such that each of the associated edge groups contains A or $G_1 = G_2$ and $g \in G_1$.*

Proof. (i) The Subgroup Theorem tells us that A is the fundamental group of a graph Γ' of groups where the associated groups are conjugates of subgroups of the groups associated to Γ. Our aim is to show that A must be a vertex group of Γ'.

The fact that $e(A) \leq 1$ tells us that each edge of Γ' is trivial and that Γ' is a tree. Thus the vertex groups of Γ' are partially ordered by inclusion. Suppose that $A_1 \subset A_2 \subset \ldots$ is an infinite ascending chain of vertex groups of Γ'. If all the inclusions are strict, then each A_i equals an edge group of Γ'. But, as Γ is a finite graph, there is an upper bound on the orders of the edge groups of Γ'. Hence, one cannot have an infinite strictly increasing chain of vertex groups of Γ'. Hence there is a maximal vertex group. This vertex group must equal A, which completes the proof of (i).

(ii) The proof of this is the same as the proof of Lemma 3.15.

Now we consider a finitely generated group G and two minimal graphs Γ and Γ' each with fundamental group G, such that each edge group is finite and each vertex group has at most one end. Note that Γ and Γ' must be finite.

Lemma 7.6. (i) There is a bijection between the vertices of Γ and Γ' such that corresponding vertex groups are conjugate in G.

(ii) Γ and Γ' have the same number of edges.

(iii) If Γ does not have distinct edges e, f with G_e lying in a conjugate of G_f, then Γ and Γ' are isomorphic as graphs and corresponding vertex or edge groups are conjugate in G.

Remarks. In (iii), the hypothesis implies that no edge group of Γ is trivial, unless Γ has only one edge.

It seems reasonable to suppose that the analogue of (i) for the edges of Γ and Γ' always holds, but we cannot prove it.

Proof. (i) Let A be a vertex group of Γ. Then $e(A) \leq 1$. Lemma 7.5 (i) tells us that A lies in a conjugate of a vertex group B of Γ'. The same lemma shows that B lies in a conjugate of a vertex group A_1 of Γ. Hence A lies in a conjugate A_1^g of A_1 for some $g \in G$. Lemma 7.5(ii) tells us that either $A = A_1$ and $g \in A$ or there is a path from A to A_1 in Γ for which each edge group contains a conjugate of A. As Γ is minimal, the second case can only occur when $A = A_1$ and the path consists of a single loop. Therefore $A = A_1^g$ and A is conjugate to B. As the groups associated to distinct vertices of Γ cannot be conjugate (because Γ is minimal), assertion (i) follows.

(ii) Let $\overline{G}$ denote the quotient of G obtained by killing all the vertex groups of Γ. This quotient is a free group of rank $E - V + 1$, where E and V are the number of edges and vertices of Γ. Part (i) tells us we obtain a group isomorphic to $\overline{G}$ by killing all the vertex groups of Γ'. Hence $E - V + 1 = E' - V' + 1$. As $V = V'$, by (i), we have $E = E'$ as required.

(iii) Let e be an edge of Γ with vertices v_1 and v_2 which may be equal. Let G_1 and G_2 be the groups associated to v_1 and v_2 and let A be the group associated to e. Then $A = G_1 \cap G_2^g$ where either $G_1 \neq G_2$ or $G_1 = G_2$ and $g \notin G_1$. It follows from Lemma 7.5 (ii), and from part (i) of this lemma, that A lies in a conjugate of an edge group B of Γ'. Similarly, B lies in a conjugate of an edge group A_1 of Γ. Our hypothesis on Γ, in (iii), implies that $A = A_1$ so that A is con-

jugate to B. Thus for each edge group of Γ, there is an edge group of Γ' conjugate to it in G and distinct edges of Γ correspond to distinct edges of Γ'. As Γ and Γ' have the same number of edges, by part (ii), we have a bijection between the edges of Γ and Γ'.

Suppose that A is an edge group of Γ and that A is contained in a conjugate of a vertex group H of Γ. Our condition that A is not contained in a conjugate of any other edge group of Γ implies that H is one of the vertex groups at the end of the edge to which A is associated. The same holds for Γ', so that the bijection between the edges of Γ and Γ' must actually induce an isomorphism of the graphs Γ and Γ'.

We turn now to another embedding result proved in [20]. The result and its proof are similar to those of Theorem 3.17, which told us that any countable group could be embedded in a 2-generator group. The result is an essential part of the proof of Theorem 7.3 for arbitrary cardinality of the groups involved.

Theorem 7.7. *If G is the fundamental group of a countable graph Γ of finite groups, where the vertex groups have bounded order, then G can be embedded in a group H_* which is the fundamental group of a finite graph of finite groups.*

Remarks. The natural homomorphism $A *_C \to \mathbb{Z}$, obtained by killing A, has kernel K equal to $\ldots *_C A *_C A *_C \ldots$. This can be seen most simply by constructing a space X whose fundamental group is K and observing that $\mathbb{Z}$ acts freely on X with quotient a space with fundamental group $A *_C$. The graph Γ corresponding to K is a copy of the real line with integer points as vertices, and all the vertex groups are copies of A, all the edge groups are copies of C. Thus $\mathbb{Z}$ acts on Γ, as a graph of groups. This is an example of how to embed a group which is the fundamental group of an infinite graph of groups into a group which is the fundamental group of a finite graph of groups. One needs a fairly uniform sort of graph Γ so that Γ admits a group action.

Proof. The aim of our proof is to work in steps so as to make Γ uniform. Since the vertex groups have bounded order we can choose a group H (for example, a symmetric group) in which they all embed.

Let G_1 be obtained from G by replacing each vertex group of Γ by a copy of H. Note that $G \subset G_1$.

Let $H_1, \ldots, H_n$ be groups, one from each isomorphism class of subgroups of H. Let $f_1, \ldots, f_N$ be the distinct embeddings of $H_1, \ldots, H_n$ in H. Then each edge of Γ_1 has a pair of f_1's associated to it. We will say that two edges of Γ_1 are <u>of the same type</u> if they have the same unordered pair associated.

We enlarge the graph of groups Γ_1 by adding countably many edges of each type joining each distinct pair of vertices of Γ_1. This new graph Γ_2 is still countable, and so is its fundamental group G_2. We have $G_1 \subset G_2$. Choose a maximal tree T in Γ_2 consisting of edges with the identity map of H at each end. Let Γ_3 be the graph of groups obtained from Γ_2 by identifying T to a point. Thus Γ_3 has one vertex labelled H and countably many loops of each type. Its fundamental group is still G_2.

We now have a graph which clearly admits a group action. Suppose that Γ_3 has m types of loop. We label the edges of Γ_3 by a_{ij}, where $1 \leq i \leq m$, and for fixed i, the suffix j runs through all the integers, thus enumerating all the loops of one given type.

G_2 has a presentation of the form

$$\{H, \{a_{ij}\} \mid a_{ij}^{-1} b_{ik} a_{ij} = c_{ik}, \text{ where } k \text{ runs through some set } K_i \text{ and } b_{ik}, c_{ik} \in H\}.$$

We define an isomorphism $\phi : G_2 \to G_2$ by $\phi(h) = h$, for $h \in H$ and $\phi(a_{ij}) = a_{i,j+1}$. This determines an extension of G_2 by $\mathbb{Z}$ which we call G_3. G_3 can be presented as

$$\{H, \{a_{i0}\}, t \mid t^{-1} h t = h \text{ for } h \in H, a_{i0}^{-1} b_{ik} a_{i0} = c_{ik} \text{ for } k \in K_i\}.$$

Hence G_3 is the fundamental group of a graph of groups which has one vertex labelled H, one loop of each type and one extra loop which has associated to it the identity map of H at each end. Hence G_3 is the required group H_*.

8. ENDS OF PAIRS OF GROUPS

The concept of the number of ends of a pair of groups (G, C), where C is a subgroup of G, is a generalization of the number of ends of a group. Recall (Section 6) the close relationship between the theory of ends of groups and the Sphere Theorem. One is also interested in conditions which will guarantee the existence of other surfaces in a 3-manifold - particularly when the fundamental group of the surface injects into the fundamental group of the 3-manifold. Thus one is interested in groups which split over infinite subgroups. The starting point of my work on ends of pairs of groups was the idea that there should be a generalization of Stallings' structure theorem to this situation. Thus one is looking for a natural definition of a number $e(G, C)$, and one hopes to prove that $e(G, C) \geq 2$ if and only if G splits over some subgroup closely related to C.

The correct definition of $e(G, C)$ is due to Houghton [33]. Recall the definition of $e(G)$. One lets PG be the power set of G, FG be the collection of finite subsets of G, each with Boolean addition, and defines $EG = PG/FG$. The right action of G on itself induces a right action of G on EG. Let $(EG)^G$ denote the subset of elements left fixed by this action (this is the same as QG/FG, where QG is as in §5). Then $e(G)$ is the dimension, as $\mathbb{Z}_2$-vector space, of $(EG)^G$.

Let C be a subgroup of G and let $H = C\backslash G$. Then we define $e(G, C)$ to be the dimension of $(EH)^G$. Clearly if C is trivial, then $e(G, C) = e(G)$. The following result justifies the claim that this is the correct definition of $e(G, C)$.

Lemma 8.1. *Let X be a finite CW-complex with a connected regular covering space $\tilde{X}$ whose covering group is G. If C is a subgroup of G, then $e(G, C) = e(C\backslash\tilde{X})$.*

Remark. The hypothesis that X is finite implies that G is f.g.

One can summarize the basic properties of $e(G, C)$ as follows.

Lemma 8.2. (i) *$e(G, C) = 0$ if and only if $|G : C|$ is finite.*
(ii) *If $G \supset G_1 \supset C$, with $|G : G_1|$ finite, then $e(G, C) = e(G_1, C)$.*

(iii) If K is a normal subgroup of G with quotient G_1 and $|K : K \cap C|$ is finite, then $e(G, C) = e(G_1, pC)$, where $p : G \to G_1$ is the natural projection.

(iv) If $C_1 \subset C \subset G$ and $|C : C_1| = n$, then $e(G, C) \leq e(G, C_1) \leq n.e(G, C)$.

These results are all the analogues of results about $e(G)$, except for (iv). One can give examples showing that either equality can be achieved in this part. The final basic property of $e(G, C)$ is

Lemma 8.3. If G splits over C, then $e(G, C) \geq 2$.

Proof. The proof is very similar to that of Lemma 6.3. We consider only the case when $G = A *_C B$. Recall the set E which was the subset of G consisting of elements whose canonical form starts in A. If $b \in B$, then $Eb = E$ and if $a \in A$, then $E \subset Ea \cup Ca \subset E \cup C \cup Ca$. The subset pE of $C\backslash G$ is left almost invariant by every element of A or B, and so is almost invariant in $C\backslash G$. Clearly pE and pE^* are infinite.

This leads us to the first large difference between $e(G)$ and $e(G, C)$. We know that $e(G)$ can only take the values 0, 1, 2 or ∞, but $e(G, C)$ can take any positive integer value. This is shown by the following example. Note that both G and C are f.g. in this example.

Example. Let F be a closed surface and let X be a compact subsurface so that no component of $F - X$ has closure homeomorphic to a 2-disc. Then the natural map $\pi_1(X) \to \pi_1(F)$ is injective, and we call the groups G and C. Now $e(G, C)$ equals the number of ends of F_C, the covering space of F with fundamental group C. But one knows that X lifts to F_C and can prove easily that F_C consists of X together with half open annuli $S^1 \times [0, \infty)$ attached to each boundary component of X. Thus $e(G, C)$ equals the number of boundary components of X. By choosing F to be of appropriately high genus, one can find pairs (G, C) for which $e(G, C)$ takes any specified value.

Originally I hoped to prove that $e(G, C) \geq 2$ if and only if G splits over some subgroup closely related to C. The following example shows

that no such result can hold.

Example. Let M be a closed, orientable irreducible 3-manifold. It can be shown that M is sufficiently large if and only if $\pi_1(M)$ splits over some subgroup. There exists such a 3-manifold which is not sufficiently large, but has a finite covering space which is sufficiently large. See [8] for a discussion of such examples. Thus we have an unsplittable group G with a subgroup G_1 of finite index which splits over some subgroup C. Hence $e(G, C) = e(G_1, C) \geq 2$, but G is unsplittable. (One can find a subgroup C which is finitely generated, so there is nothing pathological about this example.)

This example suggests that one must be content to prove that $e(G, C) \geq 2$ if and only if G has a subgroup G_1 of finite index such that G_1 splits over some subgroup closely related to C. It then seems reasonable that one will need a residual finiteness condition on G.

We say that a group G is residually finite if given $g \in G$, there is $G_1 \subset G$, such that $|G : G_1|$ is finite and $g \notin G_1$. If C is a subgroup of G, we say that G is C-residually finite if given $g \in G - C$ there is $G_1 \subset G$ such that $|G : G_1|$ is finite, $G_1 \supset C$ and $g \notin G_1$. The natural result seems to be the following, which is proved in [21].

Theorem 8.4. *If G and C are f.g. groups and G is C-residually finite, then $e(G, C) \geq 2$ if and only if G has a subgroup G_1 of finite index such that G_1 contains C and splits over C.*

The residual finiteness condition cannot be omitted.

Example. Let $G = A * C$, where A and C are infinite, simple, f.g. groups. Thus G has no subgroups of finite index and C has no subgroups or supergroups of finite index. Now for any non-trivial free product $A * C$ except for $\mathbb{Z}_2 * \mathbb{Z}_2$, it is easy to show that $e(G, C) = \infty$. But if G had a subgroup G_1 of finite index which split over some subgroup C_1 closely related to C one would be forced to have $G = G_1$ and $C_1 = C$. The example is completed by showing that G cannot split over C.

Lemma 8.5. Let $G = A * C$, where A is indecomposable and not infinite cyclic. Then G cannot split over C.

Proof. Suppose $G = X *_C Y$ or $X *_C$. As no conjugate of A meets C, we see from the Subgroup Theorem that A lies in a conjugate of X or Y. We can suppose X is involved. Use $\langle A \rangle$ to denote the normal closure of A in a group containing A. We know that $G/\langle A \rangle \cong C$. Hence we have the equations $C = X/\langle A \rangle *_C Y$ or $C = X/\langle A \rangle *_C$. The second equation is impossible, and the first equation can only hold when $X/\langle A \rangle = C = Y$. But the equation $C = Y$ contradicts the hypothesis that G splits over C.

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6 · An example of a finitely presented solvable group

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§1. Introduction

Let $R = \mathbb{Z}[\frac{1}{p}]$ be the ring of rationals with denominator a power of p, p a prime. Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ & * & * & * \\ & & * & * \\ & & & 1 \end{pmatrix}$$

with entries in R and (positive) units in the diagonal. The purpose of this note is to prove the elementary fact that G is finitely presented.

The point is that the centre Z(G) of G is not finitely generated, since isomorphic to R. So this example gives a negative answer to the following question of P. Hall [5]: Does every solvable finitely presented group satisfy the maximal condition for normal subgroups? Equivalently: Is every homomorphic image of a finitely presented solvable group itself finitely presented? There has been a failed attempt to give a counter-example ([7], s. [9]). R. Bieri and R. Strebel have recently announced results [4] implying that there cannot be a counterexample which is an extension of a class 2 nilpotent group by an abelian group. Since our example is class 3 nilpotent by abelian, it is simplest possible in this respect. In [1] Problem 7.2(i) it is asked whether every finitely presented (not necessarily solvable) group has a finitely generated centre. A counterexample (not solvable) was given in [8].

The constructions P. Hall makes with his group H of [6, p. 349] can be imitated with our G, notably G mod a cyclic subgroup of Z(G) is finitely presented solvable not Hopfian and not residually finite. In [2] Baumslag failed to give an example of a finitely presented solvable group which is not residually finite.

G is not constructable, since every homomorphic image of a constructable solvable group is finitely presented [3]. But G has a presentation similar to that of an HNN-extension: referring to our list of generators and relations for G below, one can prove that $\langle e_{12}, e_{23}, e_{34}$; relations involving them$\rangle$ is a presentation of $N(\mathbb{Z})$, the subgroup of matrices with integer entries and 1's on the diagonal. The remaining generators - i.e. d_2, d_3 - and relations define something like an HNN-extension except with $\mathbb{Z} \oplus \mathbb{Z}$ instead of $\mathbb{Z}$.

I thank R. Bieri and R. Strebel for informing me about the existing literature and problems. I hope to be able to incorporate this example into a theory of finite resp. compact presentation of arithmetic and algebraic groups.

§2. Proof

Let us say an automorphism α of a group B contracts B into a subgroup C of B if:

(1) $\alpha(C) \subset C$ and

(2) $\bigcup_{n \in \mathbb{Z}} \alpha^{-n}(C) = B$.

For every finite subset S of B we have $\alpha^n(S) \subset C$ for almost every $n \in N$.

Lemma. Let $A = \mathbb{Z} \ltimes B$ be a split extension of B with $\mathbb{Z}$. Suppose the automorphism α of B corresponding to a generator of $\mathbb{Z}$ contracts B into a finitely presented subgroup C of B. Then A is finitely presented.

In fact, if $\langle x_1, \ldots, x_n; r_1, \ldots, r_m \rangle$ is a presentation of C, express $\alpha(x_i)$ as a word w_i in $x_1, \ldots, x_n$. Then $\langle x_1, \ldots, x_n, t; r_1, \ldots, r_m, t x_1 t^{-1} w_1^{-1}, \ldots, t x_n t^{-1} w_n^{-1} \rangle$ is a presentation of A. So $A = C *_\alpha$ is an HNN-extension.

We now give a set of generators and defining relations for our group G. The generators are d_2, d_3, e_{12}, e_{23}, e_{34}.

For the relations we use the notations

$$(x, y) = x^{-1} y^{-1} x y$$
$$x^y = y^{-1} x y$$

(A) $(d_2, d_3) = 1$

(B1) $(e_{12}, e_{34}) = 1$.

For the following relations let us define

$$e_{13}^{-1} = (e_{12}, e_{23})$$
$$e_{24}^{-1} = (e_{23}, e_{34}) .$$

(B2) $(e_{13}, e_{12}) = 1$

(B3) $(e_{13}, e_{23}) = 1$

(B4) $(e_{24}, e_{23}) = 1$

(B5) $(e_{24}, e_{34}) = 1$

(B6) $(e_{13}, e_{24}) = 1$

(C 12.2) $d_2^{-1} e_{12} d_2 = e_{12}^p$

(C 12.3) $(e_{12}, d_3) = 1$

(C 23.2) $d_2 e_{23} d_2^{-1} = e_{23}^p$

(C 23.3) $d_3^{-1} e_{23} d_3 = e_{23}^p$

(C 34.2) $(e_{34}, d_2) = 1$

(C 34.3) $d_3 e_{34} d_3^{-1} = e_{34}^p$

I claim that this set of 5 generators and 13 relations is a presentat of G. Let H be the group with the above presentation. We have a hom morphism $H \to G$ defined by

$$d_2 \mapsto \begin{pmatrix} 1 & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$d_3 \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & 1 \end{pmatrix}$$

$$e_{12} \mapsto \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

etc.

The above presentation of H contains presentations of certain subgroups of G. Specifically: $\langle d_2, d_3, e_{12}, e_{34}$; (A), (B1) and the first and last two of the C-relations$\rangle$ is a presentation of the subgroup G_1 of matrices of the form

$$G_1 : \left(\begin{array}{cc|cc} 1 & * & & \\ & * & & \\ \hline & & * & * \\ & & & 1 \end{array}\right),$$

similarly $\langle d_2, d_3, e_{12}, e_{23}$; the relations involving them$\rangle$ for

$$G_2 : \left(\begin{array}{ccc|c} 1 & * & * & \\ & * & * & \\ & & * & \\ \hline & & & 1 \end{array}\right),$$

similarly $\langle d_2, d_3, e_{23}, e_{34}$; the relations involving them$\rangle$ for

$$G_3 : \left(\begin{array}{c|ccc} 1 & & & \\ \hline & * & * & * \\ & & * & * \\ & & & 1 \end{array}\right).$$

Our claim for G_1 follows directly from the lemma applied to the two blocks. For G_2 and G_3 first apply the lemma to

$$A_1 = \left\{ \begin{pmatrix} 1 & \circledast & * \\ & 1 & * \\ & & * \end{pmatrix}, \ * \in R, \ \circledast \in \mathbb{Z} \right\}$$

with B the subgroup with 1's in the diagonal and C the subgroup of B with integer entries. Then apply the lemma again for

$$A = \left\{ \begin{pmatrix} 1 & * & * \\ & * & * \\ & & * \end{pmatrix} \right\}$$

with B the subgroup with 1 as second entry on the diagonal and $C = A_1$.

Now we have inverse homomorphisms $G_i \to H$, $i = 1, 2, 3$. In particular we have in H the relations:

$$\left.\begin{array}{ll} (C\ 13.2) & (e_{13},\ d_2) = 1 \\ (C\ 13.3) & d_3^{-1} e_{13} d_3 = e_{13}^{p} \end{array}\right\} \text{ from } G_2$$

$$\left.\begin{array}{ll} (C\ 24.2) & d_2 e_{24} d_2^{-1} = e_{24}^{p} \\ (C\ 24.3) & (e_{24},\ d_3) = 1 \end{array}\right\} \text{ from } G_3$$

As a consequence we have an inverse homomorphism $G_4 \to H$,

$$G_4 : \begin{pmatrix} 1 & & * & \\ & * & * & * \\ & & * & \\ & & & 1 \end{pmatrix}$$

since $G_4 = \langle d_2,\ d_3,\ e_{13},\ e_{23},\ e_{24};$ the relations involving them including C 13.2 through C 24.3$\rangle$. The proof is as above for G_2 and G_3. The proof implies that $G_1, \ldots, G_4$ are constructable. So are all their intersections G_σ.

For every pair (i, j), $1 \leq i < j \leq 4$, $(i, j) \neq (1, 4)$, we have a homomorphism into one of $G_1, \ldots, G_4, G_k$ say:

$$R \to G_k \to H,$$

$$r \mapsto 1 + r.E_{ij}$$

where E_{ij} is the matrix with (m, n)-entry equal $\delta_{im} \cdot \delta_{jn}$. The image of r in H does not depend on k. We call it e_{ij}^r. Now all relations (A) through (C 24.3) occur in one or several of the groups G_k, so all of them hold for e_{ij}^r instead of e_{ij}; i.e.

$$(B1)' \qquad \begin{array}{l} (e_{12}^r,\ e_{34}^s) = 1 \\ (e_{12}^r,\ e_{23}^s) = e_{13}^{-rs} \end{array}$$

only depends on r.s,

$$(C\ 12.2)' \qquad d_2^{-n} e_{12}^r d_2^n = e_{12}^{p^n . r},$$

etc.

We now make use of the following formula of P. Hall:

$$(x^y, (y, z)) \cdot (y^z, (z, x)) \cdot (z^x, (x, y)) = 1 \qquad (*)$$

Computing for $x = e_{12}^r$, $y = e_{23}^s$, $z = e_{34}^t$ we obtain:

$$(e_{13}^{rs}, e_{34}^t) = (e_{12}^r, e_{24}^{st})$$

which implies:

$$(e_{13}^s, e_{34}^t) = (e_{12}^u, e_{24}^v) \quad \text{if } s \cdot t = u \cdot v$$

which we define as $e_{14}^{st} \in H$.

The next step is to see that e_{14}^r is central in G:

Since $e_{14}^r \in \langle e_{12}^r, e_{24} \rangle$ it commutes with d_3, e_{13}^s and e_{34}^t.

Since $e_{14}^r \in \langle e_{13}^r, e_{34} \rangle$ it commutes with d_2, e_{12}^s and e_{24}^t.

Now (*) with $x = e_{13}^r$, $y = e_{34}$, $z = e_{23}$ implies: $(e_{14}^r, e_{23}) = 1$, which proves our claim.

We now conclude that $H \to G$ is an isomorphism since both groups are solvable and have isomorphic factors of their derived series.

Note added in proof

Bieri [A connexion between the integral homology and the centre of a rational linear group; preprint] has shown that G is not of type $(FP)_3$. Remeslennikov has given examples of finitely presented non solvable groups with centre of infinite rank. He also pointed out that our group of matrices with entries in $R = Z[x]_S$, $S = \{x, x+1\}$, is a finitely presented solvable group having a centre of infinite rank (oral communication).

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7 · $SL_3(F_q[t])$ is not finitely presentable

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Let K be a function field in one variable over a finite field of constants, denote by S a finite non-empty set of primes of K and by $\mathcal{O}_S$ the ring of S-integers in K, and suppose that $\mathcal{G}$ is a Chevalley group. We are interested in the question, whether the group $\Gamma = \mathcal{G}(\mathcal{O}_S)$ is finitely generated or finitely presented. It is known since many years that in the 'simplest' case, the group $\Gamma = SL_2(\mathbb{F}_q[t])$ is not finitely generated (cf. [7], II. 1. 6), and that in all other cases, that means if rank $\mathcal{G} \geq 2$ or $|S| \geq 2$, the group Γ is finitely generated (cf. [1], which treats also the non-split case). The problem of finite presentability could be attacked only some years ago and there are now the following results:

(a) If $\Gamma = SL_2(\mathcal{O}_S)$, U. Stuhler showed that Γ is finitely presented if $|S| \geq 3$ and is not finitely presentable if $|S| = 2$ ([9]).

(b) If $|S| = 1$, U. Rehmann and C. Soulé proved that $GL_n(\mathcal{O}_S)$ is finitely presented for $n \geq 4$ and that for a simple Chevalley group $\mathcal{G}$ of rank $\mathcal{G} \geq 3$ the group $\Gamma = \mathcal{G}(\mathbb{F}_q[t])$ is also finitely presented ([6]).

(c) If $\mathcal{O}_S = \mathbb{F}_q[t, t^{-1}]$, J. Hurrelbrink showed that Γ is finitely presented for a simple Chevalley group $\mathcal{G}$ of rank $\mathcal{G} \geq 2$ - with the possible exception of the type G_2 ([4]).

So a reasonable conjecture for simple groups would be that Γ is finitely presented if and only if rank $\mathcal{G} + |S| \geq 4$ (and it is true that Γ is finitely generated iff rank $\mathcal{G} + |S| \geq 3$).

So in particular one needs a 'negative' result for $|S| = 1$ and rank $\mathcal{G} = 2$, this is proved here for special rings.

Theorem. If $\mathcal{G}$ is a Chevalley group of rank 2, then $\Gamma = \mathcal{G}(\mathbb{F}_q[t])$ is not finitely presentable, if we assume for type B_2 and G_2 that $(-1) \notin \mathbb{F}_q^2$.

The idea of the proof is very much the same as in Stuhler's paper; especially one has to use the operation of Γ on the corresponding Bruhat-Tits-building (all necessary results on such buildings can be found in [3]).

1. Let $\mathbb{F}_q(t)$ be the rational function field in one variable t over the finite field $\mathbb{F}_q$ and v the discrete valuation of $\mathbb{F}_q(t)$, which corresponds to the infinite point, that means $v(\frac{p}{q}) := \deg q - \deg p$ for $p, q \in \mathbb{F}_q[t]$.

Denote by k the completion of $\mathbb{F}_q(t)$ with respect to v, by o its valuation ring with prime ideal p and choose t^{-1} as prime element, such that $k = \mathbb{F}_q((\frac{1}{t}))$, the field of formal series in $\frac{1}{t}$.

Now consider a Chevalley group $\mathcal{G}$ of rank 2 (given as a scheme over Z). We may assume for our problem that $\mathcal{G}$ is simple and simply connected: If $\mathcal{G}$ is not simply connected, then there exists a covering $\tilde{\mathcal{G}} \xrightarrow{\phi} \mathcal{G}$, where $\tilde{\mathcal{G}}$ is simply connected, ϕ has finite kernel and $\phi(\tilde{\mathcal{G}}(\mathcal{O}_S))$ has finite index in $\mathcal{G}(\mathcal{O}_S)$ (for an arbitrary ring $\mathcal{O}_S$ of integers, cf. [2], Satz 1). If $\mathcal{G}$ is simply connected of rank 2, but not simple, it would be $SL_2 \times SL_2$, and $SL_2(\mathbb{F}_q[t])$ is not even finitely generated.

Let $\mathcal{T}$ denote a maximal split torus of $\mathcal{G}$ and Φ the root system of $\mathcal{G}$ with respect to $\mathcal{T}$, such that Φ is of type A_2 or B_2 or G_2. For each root $c \in \Phi$ there exists an isomorphism x_c from the additive group onto a one-dimensional root group $\mathcal{U}_c$ and an epimorphism ζ_c from SL_2 onto the subgroup of $\mathcal{G}$, generated by $\mathcal{U}_c$ and $\mathcal{U}_{-c}$ (for these facts and the following ones see [3], 6.1.3b or [5]). Define $\mathcal{U} := \prod_{c \in \Phi} \mathcal{U}_c$; $\mathcal{U}$ is the unipotent radical of the Borel subgroup $\mathcal{B} = \mathcal{T} \ltimes \mathcal{U}$ and $\mathcal{U}^- := \prod_{c \in \Phi^-} \mathcal{U}_c$ is the opposite group of $\mathcal{U}$. We set $G = \mathcal{G}(k)$ and it is well known ([5]), that G is a group with a BN-pair, where the group B is given as $\mathcal{U}^-(p)\mathcal{T}(o^*)\mathcal{U}(o)$ (o^* is the group of units in o).

If we denote now for each of our three types a fundamental system of roots by $\{a, b\}$ and the highest root by a_0 and define

$$w_a := \zeta_a\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad w_b := \zeta_b\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad w_0 := \zeta_{a_0}\left(\begin{pmatrix} 0 & t \\ -\frac{1}{t} & 0 \end{pmatrix}\right),$$

then the affine Weyl group W of the BN-pair is generated by w_a, w_b and

w_0, the linear Weyl group W_0 by w_a and w_b. We have three maximal parahoric subgroups P_i $(i = 0, 1, 2)$, where $P_0 = BW_0B = \mathcal{G}(\mathfrak{o})$, P_1 is generated by B, w_a and w_0 and P_2 is generated by B, w_b and w_0.

2. We will now look at the group $\Gamma := \mathcal{G}(\mathbb{F}_q[t])$, which is a discrete subgroup of G. In the rank 2-case ($\mathcal{G}$ is simple!) it is known that Γ is finitely generated, but we have to be more precise and will show that in case A_2 and B_2, Γ is generated by $\Gamma_0 \cup (\Gamma_1 \cap \Gamma_2)$ with $\Gamma_i := \Gamma \cap P_i$ for $i = 0, 1, 2$, whereas in case G_2, Γ is generated by $\Gamma_0 \cup \Gamma_3$ with $\Gamma_3 := \Gamma \cap P_3$ and $P_3 := w_0 P_0 w_0^{-1}$.

This can easily be done by the help of Chevalley's commutator formula:

$$[x_c(u),\ x_d(v)] = \prod_{pc+qd} x_{pc+qd}\,(C_{c,d;p,q}\, u^p v^q),$$

where $c, d \in \Phi$, but $d \neq -c$, $p, q \in \mathbb{N}$, $pc + qd \in \Phi$, $C_{c,d;p,q} \in \mathbb{N}$, $u, v \in k$.

Starting with elements $x_c(t^r)$ and $x_d(t^s)$ with small positive exponents r and s, this formula provides us with elements $x_e(t^n)$ with a higher exponent n ($e \in \Phi$), and - iterating this process - we get finally elements with arbitrary high powers of t.

We have to use the following facts:

(i) Γ is generated by its subgroups $\Gamma_c := \mathfrak{U}_c(\mathbb{F}_q[t])$ $(c \in \Phi)$, because $\mathbb{F}_q[t]$ is euclidean.

(ii) $\mathfrak{U}_c(\mathbb{F}_q)$ and W_0 are contained in $\Gamma_0 = \Gamma \cap \mathcal{G}(\mathfrak{o}) = \mathcal{G}(\mathbb{F}_q)$.

(iii) Γ_1 and Γ_2 contain the element $x_{a_0}(t)$.

Because the groups $\mathfrak{U}_c$ are conjugate under W_0 for all c of the same root length, it is enough to construct for each $n \in \mathbb{N}$ an element $x_c(t^u)$ only for one representative c of each root length (we neglect the constant factors). Let us examine the three cases. For type A_2 there exists only one root length, furthermore Γ_1 and Γ_2 contain the element $x_{a+b}(t)$ and we have

$$[x_a(t^r),\ x_b(t^s)] = x_{a+b}(t^{r+s}).$$

In the case B_2 we have the short positive roots a and a + b whereas

b and $2a + b$ are the long ones; the highest root is $2a + b$. The formula $[x_a(1), x_b(t^n)] \cdot x_{2a+b}(t^n)^{-1} = x_{a+b}(t^n)$ gives us the element $x_{a+b}(t^n)$, when $x_{2a+b}(t^n)$ and therefore also $x_b(t^n)$ are already available, finally - starting with $n = 0$ - $[x_a(t), x_b(t^n)] \cdot x_{a+b}(t^{n+1})^{-1} = x_{2a+b}(t^{n+2})$ gives the higher exponents.

In the case G_2 there exist three short positive roots, namely a, $a+b$, $2a+b$ and three long positive roots b, $3a+b$, $3a+2b$, the highest root is $3a+2b$, moreover the long roots form a system of type A_2.

Γ_3 contains the elements $x_{a+b}(t)$, $x_{2a+b}(t)$, $x_b(t)$, $x_{3a+b}(t)$ (and $x_{3a+2b}(t^2)$) - to see this, look for instance at Figure 3. As in the case A_2 we obtain $x_b(t^n)$ with arbitrary n, using only the long roots. Then we have the formula

$$[x_a(t^r), x_b(t^s)] = x_{a+b}(t^{r+s}) x_{2a+b}(-t^{2r+s}) x_{3a+b}(-t^{3r+s}) x_{3a+2b}(t^{3r+2s})$$

which yields $x_{2a+b}(t^n)$ by iteration, starting with $r = 1$, $s = 0$.

3. In order to examine the question of finite presentability of Γ we now look at the Bruhat-Tits-building $\mathcal{J}$, which can be assigned to the BN-pair of the group G mentioned above (for the details of $\mathcal{J}$ see [3]). As a fundamental chamber we use the 2-simplex C_0 with vertices P_i ($i = 0, 1, 2$), such that $\mathcal{J} = G \cdot C_0$, where G operates on $\mathcal{J}$ from the left by inner automorphisms: $P \to g P g^{-1}$ for each parahoric group P. The standard apartment $A = W \cdot C_0$ is a real plane in our three cases, the standard quarter Q is defined as $\{x \in A \mid c(x) \geq 0 \text{ for all } c \in \Phi^+\}$ (where the roots c are considered as elements of V^*, V the underlying vector space of A). As a subgroup of G, Γ operates also on $\mathcal{J}$ and Q is a simplicial fundamental domain for the action of Γ on $\mathcal{J}$ as was shown by Soulé ([8], thm. 1). For each $n \in \mathbb{N}$ we define now $Q(n) := \{x \in Q \mid a_0(x) \leq n\}$, where a_0 is the highest root of Φ and set $\mathcal{J}(n) := \Gamma \cdot Q(n)$; so we have $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}_n$.

We denote by E the system of generators which we described in 2 for the different cases and consider now relations in E. Since Γ_i stabilizes the vertex P_i, we can adjoin with each word $e_1 e_2 \ldots e_r$ ($e_i \in E$) an edge-path in $\mathcal{J}$, which is given by the following sequence of edges:

P_0P, $e_1(P_0P)$, $e_1e_2(P_0P)$, ..., $e_1e_2 \dots e_r(P_0P)$, where P is the vertex P_1 or P_2 in case A_2 or B_2 and $P = P_3$ in case G_2 (here the 'edge' P_0P_3 is the union of two edges of $\mathcal{J}$, cf. Figure 3). We should remark that in the first two cases the path with $P = P_1$ is homotopic to that with $P = P_2$. By this construction relations correspond to closed edge-paths which start and end in P_0. All these edge-paths are contained in $\Gamma \cdot C_0 \subseteq \Gamma \cdot Q_1 = \mathcal{J}(1)$ in the cases A_2 and B_2 and in $\mathcal{J}(2)$ in case G_2.

We now proceed in the same manner as Stuhler did in the case of $SL_2(\mathcal{O}_S)$ for $|S| = 2$ (see [9]). Assume that Γ can be defined by a finite number of relations $r_1, \dots, r_q$ (viewed as words in E). Since $\mathcal{J}$ is contractible, the corresponding paths $\bar{r}_1, \dots, \bar{r}_p$ can be contracted to P_0; in any such contraction of a path $\bar{r}_i$ only finitely many simplices are involved. So there exists a number N such that each of the paths $\bar{r}_i$ can be contracted in $\mathcal{J}(N)$. By our assumption this is true for each path $\bar{r}$ that comes from an arbitrary relation r: r is the product of conjugates of the relations r_i by elements of Γ, to fix the ideas assume that

$$r = g_1 r_{i_1} g_1^{-1} \cdot g_2 r_{i_2} g_2^{-1} \cdot \dots \cdot g_q r_{i_q} g_q^{-1} \quad \text{with } g_1, \dots, g_q \in \Gamma .$$

Write the element g_j as a word in E and denote the corresponding path from P_0 to $g_j(P_0)$ by $\bar{g}_j$, the inverse path by $\bar{g}_j^{-1}$, then the path adjoined with r is

$$\bar{r} = (\bar{g}_1 \circ g_1(\bar{r}_{i_1}) \circ \bar{g}_1^{-1}) \circ \dots \circ (\bar{g}_q \circ g_q(\bar{r}_{i_q}) \circ \bar{g}_q^{-1}).$$

Each $\bar{r}_{i_j}$ is contractible in $\mathcal{J}(N)$, and because $\mathcal{J}(N)$ is Γ-invariant, the same is true for $g_j(\bar{r}_{i_j})$ and therefore for the whole path $\bar{r}$. If we want to prove that Γ cannot be finitely presented, we have to show that for every natural number n there is a relation r whose corresponding path is not contractible in $\mathcal{J}(n)$.

So we are left to construct such relations in each of our three cases.

Case A_2: $G = SL_3(k)$.

Let us begin with the commutator formula

$$x_{a+b}(t^m) = [x_a(t),\ x_b(t^{m-1})] = [w_b^{-1} x_{a+b}(-t) w_b,\ w_a^{-1} x_{a+b}(t^{m-1}) w_a];$$

in the last term we can substitute the analogous expression for $x_{a+b}(t^{m-1})$ and the inverse one for $x_{a+b}(t^{m-1})^{-1}$ and get by iteration $x_{a+b}(t^m)$ as a product p_1 of elements $w_a^{\pm 1}$, $w_b^{\pm 1}$, $x_{a+b}(t)^{\pm 1} \in E$. To p_1 we can assign a path $\bar{p}_1$ which starts in P_0 with edge P_0P_1 and ends in $x_{a+b}(t^m)P_0$. Symmetric to this procedure we also have

$$x_{a+b}(t^m) = [x_b(t),\ x_a(-t^{m-1})] = [w_a x_{a+b}(-t)w_a^{-1},\ w_b^{-1}x_{a+b}(t^{m-1})w_b]$$

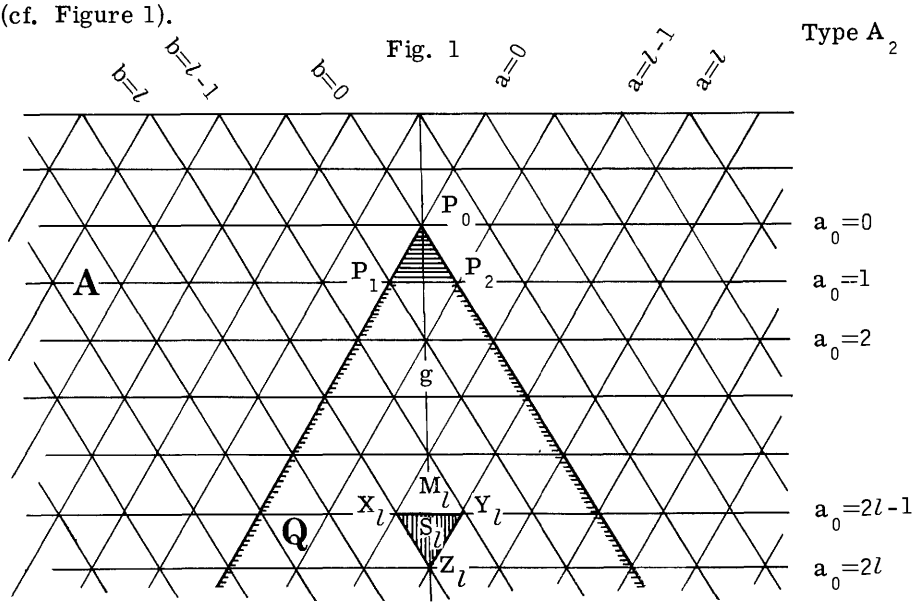
and we get by iteration $x_{a+b}(t^m)$ as a product p_2 of elements in E, and we can adjoin with p_2 the path $\bar{p}_2$, which starts in P_0 with edge P_0P_2 and ends also in $x_{a+b}(t^m)P_0$.

$r_m = p_1 p_2^{-1}$ equals 1 in G and is therefore a relation in E with associated path $\bar{r}_m = \bar{p}_1 \circ \bar{p}_2^{-1}$ (up to homotopy it doesn't matter that we used for the first half the images of the edge P_0P_1, for the second one the images of P_0P_2).

The transition from p_1 to p_2 was given by the substitutions $w_a \mapsto w_b$, $w_b \mapsto w_a^{-1}$, $x_{a+b}(u) \mapsto x_{a+b}(u)$ which define an automorphism σ of $SL_3(k)$ (it is the diagram-automorphism followed by the inner automorphism with $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$).

σ induces a simplicial automorphism $\bar{\sigma}$ of the Bruhat-Tits-building; the restriction of $\bar{\sigma}$ to the standard apartment A is the symmetry with respect to the line g through P_0 and the centre of the edge P_1P_2, (cf. Figure 1).

Fig. 1



We will now prove that $\bar{r}_m$ is not contractible in $\mathcal{J}(n)$ for an even m with $m > n$. For this purpose we consider for $l \in \mathbb{N}$ the triangle Δ_l in A which has the vertices X_l, Y_l, Z_l given by

$$a(Y_l) = a(Z_l) = b(X_l) = b(Z_l) = l, \quad a(X_l) = b(Y_l) = l - 1,$$

which implies $a_0(Z_l) = 2l$, $a_0(X_l) = a_0(Y_l) = 2l - 1$, in particular Z_l is on the line g, X_l and Y_l are symmetric with respect to g.

We delete the interior of Δ_l and define $\mathcal{J}_l := \mathcal{J} \setminus \mathring{\Delta}_l$, so $\mathcal{J}_l \subseteq \mathcal{J}(n)$ if $n \leq 2l - 1$. Since $\mathcal{J}$ is contractible along geodesic lines ([3], 2.5), there is a contraction of $\mathcal{J}$ to each point of $\mathcal{J}$, we use for this point the barycentre S_l of Δ_l and denote by $\rho : \mathcal{J}_l \to \partial\Delta_l$ the map induced by the retraction from $\mathcal{J}$ to S_l.

If we can show that $\rho(P_0) = M_l$, where M_l is the centre of $X_l Y_l$ and $\rho(R_m) = Z_l$, where $R_m = x_{a+b}(t^m)P_0$, then we see that $\rho(\bar{p}_1)$ is a path in $\partial\Delta_l$ from M_l to Z_l and also $\rho(\bar{p}_2)$ is a path in $\partial\Delta_l$ from M_l to Z_l, but symmetric to $\rho(\bar{p}_1)$ with respect to g. Therefore $\rho(\bar{r}_m) = \rho(\bar{p}_1 \circ \bar{p}_2^{-1})$ is a closed path in $\partial\Delta_l$ and not homotopic to zero, from this it follows that $\bar{r}_m$ is not contractible in $\mathcal{J}_l$, which is contained in $\mathcal{J}(n)$ for $n \leq 2l - 1$.

So we have to compute $\rho(P_0)$ and $\rho(R_m)$ and that means we have to determine the geodesic lines between P_0 and S_l and between R_m and S_l. This first problem is trivial because P_0 and S_l are contained in A, their geodesic is a segment of g and thus we have $\rho(P_0) = M_l$. In the second case we assert that the geodesic between P_0 and R_m contains the segment $S_l Z_l$ of g, if we choose $m \geq 2l$. To prove this we construct a minimal gallery between P_0 and R_m using the Bruhat decomposition of $x_{a+b}(t^m)$, for convenience we choose m even:

$$x_{a+b}(t^m) = \begin{pmatrix} 1 & t^m \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ t^{-m} & 1 \end{pmatrix} \begin{pmatrix} & t^m \\ -t^{-m} & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ t^m & 1 \end{pmatrix}.$$

Here $b := \begin{pmatrix} 1 & \\ t^{-m} & 1 \end{pmatrix}$ is in B, we define $d := \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ and have

$$w_{a+b} = w_a w_b w_a^{-1} = \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}, \quad w_0 = \begin{pmatrix} & & t \\ & 1 & \\ -t^{-1} & & \end{pmatrix} = d w_{a+b},$$

so we get

$$x_{a+b}(t^m) = b \cdot d^m \cdot w_{a+b} \cdot b = b(dw_{a+b}w_{a+b}^{-1}) \ldots (dw_{a+b}w_{a+b}^{-1})\ dw_{a+b} \cdot b$$
$$= b\ (w_0 w_{a+b}^{-1})^{m-1} w_0 b.$$

We observe that $w_{a+b}w_0^{-1}b \cdot w_0 w_{a+b}^{-1} = \begin{pmatrix} 1 & & \\ t^{m-2} & 1 & \\ & & 1 \end{pmatrix}$, thus with the definition $b_0 := \begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix}$ and $m \geq 2l$, m even, we obtain the formula

$$x_{a+b}(t^m) = (w_0 \cdot w_{a+b}^{-1})^{l-1} w_0 b_0 w_{a+b}^{-1} (w_0 w_{a+b})^{m-l-1} w_0 b. \qquad (*)$$

It is easy to see that $w_0 w_a w_b^{-1} w_a^{-1} w_0 w_a w_b^{-1} w_a^{-1} \ldots w_0 w_a w_b^{-1} w_a^{-1} w_0$ is a reduced decomposition of a word w in W, if one looks at the reflections in A, defined by these elements. With the sections of this product we define a sequence of chambers in A, namely C_0, $w_0 C_0$, $w_0 w_a C_0, \ldots, wC_0$ and this is a minimal gallery in A between C_0 and wC_0. By all these partial products of w the point P_0 is mapped on a point of the line g, especially we have $(w_0 w_{a+b}^{-1})^{l-1} w_0 C_0 = \Delta_l$ and the geodesic line between P_0 and wP_0 is contained in g.

From this minimal gallery between C_0 and wC_0, we obtain a minimal gallery between C_0 and $x_{a+b}(t^m)C_0$, if we multiply it with b from the left side, because $x_{a+b}(t^m) = bwb$ and $bC_0 = C_0$, since the stabilizer of C_0 is B. The formula $(*)$ implies that b operates trivially on the part of the gallery between C_0 and $(w_0 w_{a+b}^{-1})^{l-1} C_0$, in particular the line segment of g between P_0 and Z_l is also part of the geodesic line between P_0 and $x_{a+b}(t^m)P_0 = R_m$ and thus $S_l Z_l$ is contained in this geodesic line. From this we conclude finally that $\rho(R_m) = Z_l$. Thus for a given $n \in \mathbb{N}$ we can take for the wanted relation r, which is not contractible in $\mathcal{G}(n)$, the relation r_m for $m = n + 1$ or $m = n + 2$, m even.

Case B_2: $G = Sp_4(k)$.

We start with the commutator formula

$$x_{2a+b}(t^m) = [x_a(t),\ x_b(t^{m-2})].\, x_{a+b}(-t^{m-1})$$

and write the factors on the right side as products of elements in E and $x_{2a+b}(t^j)^{\pm 1}$ with $j < m$:

$$x_{2a+b}(t^m) = [w_b x_{a+b}(t) w_b^{-1},\ w_a x_{2a+b}(-t^{m-2}) w_a^{-1}].\, x_{2a+b}(t^{m-1}).\, [x_b(t^{m-1}), x_a(t$$

$$= [w_b x_{2a+b}(t) [w_a x_{2a+b}(-t) w_a^{-1},\ x_a(1)] w_b^{-1},\ w_a x_{2a+b}(-t^{m-2}) w_a^{-1}].$$

$$\cdot\ x_{2a+b}(t^{m-1}).\, [w_a x_{2a+b}(-t^{m-1}) w_a^{-1},\ x_a(1)].$$

In the last term we can substitute for $x_{2a+b}(\pm t^{m-1})$ and $x_{2a+b}(\pm t^{m-2})$ the analogous expression and obtain by iteration $x_{2a+b}(t^m)$ as a product p_1 of elements in E.

To get from this product a 'symmetric' relation, we apply the inner automorphism σ with w_b, so we have:

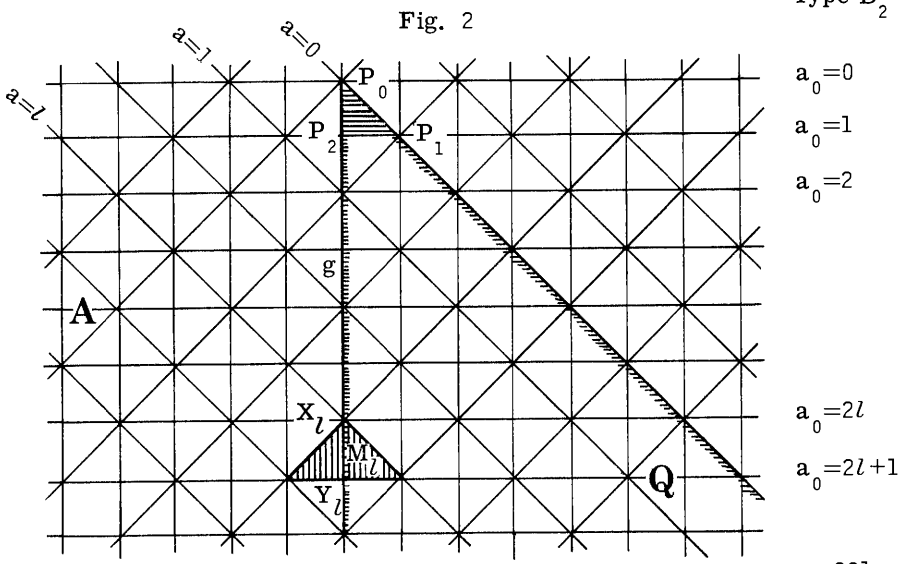
$$x_{2a+b}(t^m) \overset{\sigma}{\mapsto} x_{2a+b}(t^m),\ w_a \overset{\sigma}{\mapsto} w_{a+b}^{-1},\ w_b \overset{\sigma}{\mapsto} w_b,\ x_a(1) \overset{\sigma}{\mapsto} x_{a+b}(-1).$$

If we define $p_2 := \sigma(p_1)$, then $r_m := p_1 . p_2^{-1}$ is a relation in E, and again we associate with r_m an edge-path $\bar{r}_m$ in $\mathcal{J}$, which starts in P_0 with the edge P_0P_2.

σ induces a simplicial automorphism $\bar{\sigma}$ of $\mathcal{J}$, which fixes the line g through P_0 and P_2 and whose restriction to A is the symmetry with respect to g.

Type B_2

Fig. 2



This time we have to delete a segment $X_l Y_l$ of g (for some $l \in \mathbb{N}$), which is an image of $P_0 P_2$, and of course also the interior of all triangles that have $X_l Y_l$ as an edge. Denote by M_l the centre of $X_l Y_l$, by L_l the link of M_l in $\mathcal{J}$, consisting of a bunch of broken line-segments which connect X_l and Y_l, by $st(M_l)$ the open star of M_l, and define $\mathcal{J}_l := \mathcal{J} \setminus st(M_l)$. We consider once more the contraction of $\mathcal{J}$ to the point M_l and the map $\rho : \mathcal{J}_l \to L_l$ induced by this contraction. It is obvious that $\rho(P_0) = X_l$ and a similar computation as for case A_2 shows that $\rho(R_m) = Y_l$ with $R_m := x_{2a+b}(t^m)P_0$ for m big enough (depending on l).

It remains to prove, that $\rho(\bar{r}_m)$ is not homotopic to zero in L_l - and therefore $\bar{r}_m$ cannot be contractible in $\mathcal{J}_l$, which contains $\mathcal{J}(n)$ for small n (relative to l). Due to the symmetry of $\bar{r}_m$ it suffices to show that none of the broken line-segments between X_l and Y_l is invariant under $\bar{\sigma}$. Such a segment includes two edges, say $X_l Z_l$ and $Z_l Y_l$. Denote by d an element in G, which maps $P_0 P_2$ onto $X_l Y_l$, we can choose $d \in T(k)$, such that $w_b d = d w_b$. Then we have $Z_l = dPd^{-1}$, where $P_0 P_2 P$ is a 2-simplex of $\mathcal{J}$. That Z_l is invariant under $\bar{\sigma}$ means that w_b normalizes the parahoric group Z_l; since parahoric groups are identical with their normalizers, this implies that w_b is contained in Z_l, and with regard to $w_b d = d w_b$, even that $w_b \in P$. Thus the whole 2-simplex is fixed under w_b and an easy computation shows that this is possible if and only if $(-1) \in \mathbb{F}_q^2$.

Case G_2.

In this case E contains the elements $x_b(t)$ and $x_{3a+b}(t)$ and we have the formula

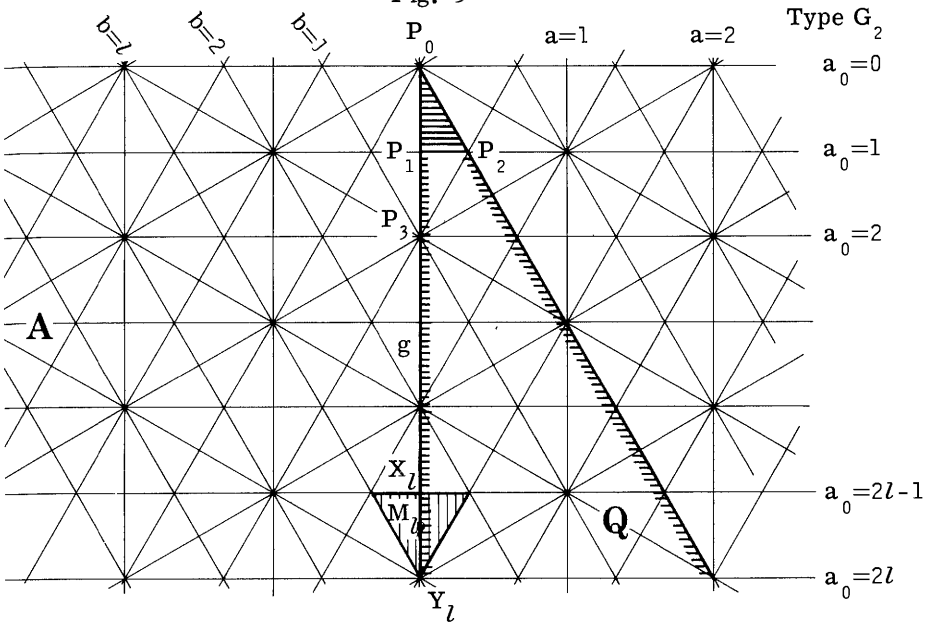
$$x_{3a+2b}(t^m) = [x_b(t),\ x_{3a+b}(t^{m-1})] = [x_b(t),\ w_b x_{3a+2b}(t^{m-1}) w_b^{-1}]$$

which can be iterated and gives finally $x_{3a+2b}(t^m)$ as a product p_1 of factors $x_b(t)^{\pm 1}$, $x_{3a+b}(t)^{\pm 1}$, $w_b^{\pm 1}$.

We obtain a second product p_2 by applying the inner automorphism σ with the element w_a, which leaves $x_{3a+2b}(t^m)$ invariant (and also the system E of generators).

Thus we have a relation $r_m = p_1 . p_2^{-1}$ and a corresponding path $\bar{r}_m$, starting with P_0P_3 which is symmetric with respect to the automorphism $\bar{\sigma}$ of $\mathcal{J}$ induced by σ.

Fig. 3



The choice of the map ρ is the same as in case B_2: Take an image X_lY_l of P_0P_1 on the axis g, denote by M_1 the centre of X_lY_l, consider the retraction of $\mathcal{J}$ to M_l and the induced map $\rho : \mathcal{J}_l \to L_l$, where $\mathcal{J}_l := \mathcal{J} \setminus st(M_l)$ and L_l is the link of M_l in $\mathcal{J}$. The same arguments as in case B_2 apply and it follows that $\bar{r}_m$ is not homotopic to zero in $\mathcal{J}_l$ or in $\mathcal{J}(n)$ for a suitable n.

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8 · Two-dimensional Poincaré duality groups and pairs

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1. Poincaré duality groups

A Poincaré duality group of dimension n in the sense of [1] (in short a PD^n-group) is a group G, acting on the infinite cyclic group $\mathbb{Z}$, such that one has natural isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbb{Z} \otimes A)$$

for all integers k and all G-modules A, ($\mathbb{Z} \otimes A$ = tensor product over $\mathbb{Z}$ with diagonal G-action). These Poincaré duality groups coincide [2] with those considered by Johnson-Wall [6], except that in general we do not assume finite presentation.

The only PD^0-group is the trivial group 1, and the only PD^1-group is infinite cyclic. Closed surfaces of genus ≥ 1 being aspherical their fundamental groups, the 'surface groups', are PD^2; we call them geometric PD^2-groups. As no other examples are available it has been conjectured that all PD^2-groups are geometric. PD^2-groups G have in fact many properties in common with surface groups: All subgroups of infinite index in G are free (Strebel [8]); if G is not perfect then it is residually nilpotent (Dyer-Vasquez [5]). J. Cohen [4] has verified the conjecture under the assumption that G admits a finite free resolution and that G/G' can be generated by 2 elements.

2. Poincaré duality pairs

The geometric analogy suggests studying PD^n-groups by trying to break them into smaller pieces and investigating 'PD^n-groups with boundary'. For the group-theoretic definition of these objects one needs relative (co)homology groups for pairs (G, S). Such a pair consists of

a group G and a family of subgroups $\underline{S} = \{S_i \leq G\} \neq \emptyset$. We take the theory considered, e.g., by Trotter [9]: The free Abelian group $\mathbb{Z}G/\underline{S} = \oplus_i \mathbb{Z}G/S_i$ over all cosets xS_i is a G-module and has an obvious 'augmentation' map $\varepsilon : \mathbb{Z}G/\underline{S} \twoheadrightarrow \mathbb{Z}$; let Δ be the kernel of ε. Then one puts, for any G-module A and $k \in \mathbb{Z}$,

$$H^k(G, \underline{S}; A) = \mathrm{Ext}_G^{k-1}(\Delta, A), \quad H_k(G, \underline{S}; A) = \mathrm{Tor}^G_{k-1}(\Delta, A).$$

Now a Poincaré duality pair of dimension n (in short a PD^n-pair) is a pair $(G, \underline{S})$, $S \neq \emptyset$, together with a G-action on $\mathbb{Z}$, such that one has natural isomorphisms

(i) $\quad H^k(G; A) \cong H_{n-k}(G, \underline{S}; \mathbb{Z} \otimes A)$

and

(ii) $\quad H^k(G, \underline{S}; A) \cong H_{n-k}(G; \mathbb{Z} \otimes A)$,

for all k and A. One can show that the two conditions (i) and (ii) imply each other, and imply that $\underline{S}$ is a finite family of PD^{n-1}-groups. Moreover, using results of [2] one obtains the following simple criterion.

PD^n-Criterion. $(G, \underline{S})$ is a PD^n-pair if and only if (a) the G-module Δ admits a finite projective resolution and (b) $H^k(G, \underline{S}; \mathbb{Z}G) = 0$ for $k \neq n$, $= \mathbb{Z}$ for $k = n$.

3. PD^2-pairs

Examples of PD^2-pairs are, of course, obtained by taking the fundamental group of a punctured closed surface together with the family of infinite cyclic subgroups generated by small circles around the points which have been removed. In this way we obtain the geometric PD^2-pairs $(G, \underline{S})$. In the orientable case, G is the free group freely generated by $x_1, \ldots, x_g, y_1, \ldots, y_g, t_1, \ldots, t_{m-1}$, $(g \geq 0,\ m \geq 1)$ and

$$\underline{S} = \{\langle t_1 \rangle, \ldots, \langle t_{m-1} \rangle, \langle t_1 t_2 \ldots t_{m-1} [x_1, y_1] \ldots [x_g, y_g] \rangle\}.$$

In the non-orientable case G is freely generated by $z_1, z_2, \ldots, z_g$,

$t_1, \ldots, t_{m-1}$, $(g \geq 1)$ and

$$\underline{S} = \{\langle t_1 \rangle, \ldots, \langle t_{m-1} \rangle, \langle t_1 t_2 \ldots t_{m-1} z_1^2 \ldots z_g^2 \rangle\}.$$

We shall now discuss the question whether all PD^2-pairs are in fact geometric. We first notice that for $n = 2$ the PD^n-criterion can be considerably simplified. If $(G, \underline{S})$ is a PD^2-pair then G is finitely generated, and of cohomology dimension ≤ 1 by (i). Hence G is free of finite rank, and $\underline{S}$ is a finite family of infinite cyclic subgroups. This implies, conversely, that Δ has a finite projective resolution of length 2. Moreover, a relative version of Stallings' theorem shows that $H^2(G, \underline{S}; \mathbb{Z}G) = \mathbb{Z}$ implies $H^1(G, \underline{S}; \mathbb{Z}G) = 0$, so that we obtain the

PD^2-Criterion. *$(G, \underline{S})$ is a PD^2-pair if and only if G is a free group of finite rank k, $\underline{S}$ a finite family of m infinite cyclic subgroups, and $H^2(G, \underline{S}; \mathbb{Z}G) = \mathbb{Z}$.*

From now on let $G = F$ be free of rank k, and $\underline{S}$ the finite family of infinite cyclic subgroups of F generated by $r_1, r_2, \ldots, r_m \in F$ respectively.

Theorem 1. *If $(F, \underline{S})$ is a PD^2-pair then there are free generators $y_1, y_2, \ldots, y_k$ of F such that r_j is conjugate to y_j for $1 \leq j \leq m-1$. In particular $m-1 \leq k$.*

Proof. First one shows that if $\underline{S}$ is the disjoint union of two non-empty families $\underline{S}_\alpha$, $\alpha = 1, 2$, and $(G, \underline{S})$ is a PD^n-pair, then $H^n(G, \underline{S}_\alpha; M) = 0$ for all G-modules M, i.e., the pairs $(G, \underline{S}_\alpha)$ are of cohomology dimension $\leq n-1$. For this we consider the short exact sequence of G-modules

$$\Delta_1 \oplus \Delta_2 \overset{\phi}{\rightarrowtail} \Delta \overset{\psi}{\twoheadrightarrow} \mathbb{Z}$$

where $\Delta = \ker(\mathbb{Z}G/\underline{S} \twoheadrightarrow \mathbb{Z})$, $\Delta_\alpha = \ker(\mathbb{Z}G/\underline{S}_\alpha \twoheadrightarrow \mathbb{Z})$, ϕ = inclusion, and ψ is given by $\psi(xS_i) = 1$ or 0 according to whether $S_i \in \underline{S}_1$ or $S_i \in \underline{S}_2$. If $(G, \underline{S})$ is a PD^n-pair, then G is a duality group of dimension $n-1$ with dualizing module $\mathbb{Z} \otimes \Delta$ (cf. [3]); hence one has the commuting diagram

$$\begin{array}{ccc} H^{n-1}(G;\, M) & \xrightarrow{\psi^*} & H^{n-1}(G;\, \mathrm{Hom}(\Delta,\, M)) \\ \downarrow \cong & & \downarrow \\ H_0(G;\, \mathbb{Z} \otimes \Delta \otimes M) & \longrightarrow & H_0(G;\, \mathbb{Z} \otimes \Delta \otimes \mathrm{Hom}(\Delta,\, M)) \end{array}$$

The evaluation map induces $H_0(G;\, \mathbb{Z} \otimes \Delta \otimes \mathrm{Hom}(\Delta,\, M)) \cong H_0(G;\, \mathbb{Z} \otimes M)$. It is easy to check that under this isomorphism the bottom map coincides with $H_0(G;\, \mathbb{Z} \otimes \psi \otimes M)$ which is surjective. Thus ψ^* is surjective, and from the long exact coefficient sequence we get

$$\bigoplus_\alpha H^{n-1}(G;\, \mathrm{Hom}(\Delta_\alpha,\, M)) = \bigoplus_\alpha H^n(G,\, \underline{S}_\alpha;\, M) = 0.$$

In the case $n = 2$ the pairs $(G,\, \underline{S}_\alpha)$ are thus of cohomology dimension ≤ 1. By a result of C. T. C. Wall [10], there exists a free basis of G which up to conjugacy contains the generators of the $S_i \in \underline{S}_1$. Assuming $m \geq 2$ and taking $\underline{S}_2 = \{S_m\}$ the assertion follows.

4. Applications of Theorem 1

(1) The PD^2-pair $(F,\, \underline{S})$ with $k = 2$ and $m = 3$ is geometric. For one is reduced to the case $F = \langle x,\, y;\, - \rangle$, $\underline{S} = \{\langle x \rangle,\, \langle y \rangle,\, \langle r \rangle\}$. But Theorem 1 also asserts that x together with a conjugate of r form a basis of F, hence we may assume $r = x^\alpha y$. The same argument applied to y and $x^\alpha y$ yields $\alpha = 1$.

(2) Let $(F,\, \underline{S})$ be a PD^2-pair with arbitrary k and $m \leq k + 1$, and take PD^2-pairs $(F_i,\, T_i)$, $1 \leq i \leq m$, corresponding to punctured tori; i. e. $F_i = \langle a_i,\, b_i;\, - \rangle$ and $T_i = \langle [a_i,\, b_i] \rangle$. Then consider the free product of F with the F_i's amalgamated along $r_i = [a_i,\, b_i]$,

$$H = \langle F,\, a_i,\, b_i;\, r_i = [a_i,\, b_i] \quad (1 \leq i \leq m) \rangle.$$

One can show ([3], Theorem 8.1) that $(F,\, \underline{S})$ is a PD^2-pair if and only if H is a PD^2-group. Moreover, if $(F,\, \underline{S})$ is geometric, so is of course H. But the converse holds! For this one has to show that if H is the fundamental group of a closed surface $\mathfrak{F}$ then there are simple closed curves on $\mathfrak{F}$ such that cutting along those realizes geometrically the given

amalgamated product decomposition. Now by Theorem 1, we can always arrange that H has a single defining relation; this shows that the absolute PD^2-conjecture for one-relator groups implies the PD^2-conjecture for all pairs.

5. Further information on a PD^2-pair (F, S) is obtained by considering the quotient of F modulo the normal subgroup generated by S, i.e. the group

$$Q = \langle F; r_1 = r_2 = \ldots = r_m = 1 \rangle .$$

By Theorem 1 we know already that Q has a presentation with $k - m + 1$ generators and a single defining relator.

Theorem 2. *The minimal number of generators for Q is $d(Q) = k - m + 1$. Moreover, if $d(Q) = 1$ then $Q = \mathbb{Z}/2\mathbb{Z}$ and if $d(Q) > 1$ then Q is a PD^2-group.*

For the proof we refer to [3]. As an application, we consider the case $m = 1$. Then $d(Q) = k$, and $Q = \langle F; S \rangle$ is itself a one-relator presentation. If $k \geq 2$ then, by Theorem 2, Q is a PD^2-group. If Q is a surface group then, by results of Zieschang and Rosenberger (cf. [7]) there is a basis of F such that the relator r has the canonical form, and hence the pair $(F; S)$ is geometric. Using the fact that one-relator presentations are aspherical together with J. Cohen's result for the case when $d(Q/Q') \leq 2$, we obtain the result that *every PD^2-pair (F, S) with $k = 2$ and $m = 1$ is geometric.*

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9 · Metabelian quotients of finitely presented soluble groups are finitely presented

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1. The purpose of this note is to announce some progress in the theory of finitely presented soluble groups. In particular, we solve Gilbert Baumslag's problem [1], [2] of discerning which finitely generated metabelian groups are finitely presented and we prove the result stated in the title. This gives a positive answer to a weak version of Philip Hall's old problem [5] whether all homomorphic images of a finitely presented soluble group be finitely presented. Moreover, this answer is sharp, for H. Abels' example (cf. Remark 1) shows that 3-step soluble images need not be finitely related.

2. Recall that an abelian group Q is orderable if there is a subset $Q^+ \subset Q$ with the following three properties

(i) Q^+ is a submonoid of Q,

(ii) $Q = Q^+ \cup Q^-$, where $Q^- = (Q^+)^{-1}$,

(iii) $Q^+ \cap Q^- = 1$.

If Q is orderable, then every subset $Q^+ \subset Q$ satisfying (i)-(iii) shall be termed an ordering of Q.

From now on Q will denote a free Abelian group of finite rank n. Then it is well known that Q is orderable. We shall consider Q-modules A with the following property:

Definition. We say that A has property (*) if, for every ordering Q^+ of Q, A is finitely generated either as a $\mathbb{Z}Q^+$-module or as a $\mathbb{Z}Q^-$-module (or both).

It is clear that (*) is inherited by all homomorphic images of A. One can show that all orderings of Q can be described in terms of flags of closed half spaces in $\mathbb{R}^n = Q \otimes_{\mathbb{Z}} \mathbb{R}$, and this description allows one to verify property (*) in specific situations.

3. Our main results can now be stated as follows:

Theorem A. Let G be a soluble group containing a normal subgroup $N \lhd G$ such that the quotient $Q = G/N$ is free Abelian. If G is finitely presented then the Q-module N/N' has property (*).

Theorem B. Let Q be a free Abelian group of finite rank and A a Q-module with property (*). Then every extension of A by Q is finitely presented.

As finite presentability is not affected when we pass to a subgroup of finite index the conjunction of Theorems A and B yields a necessary and sufficient condition for a metabelian group G to be finitely presented. Somewhat surprisingly this condition does not involve the extension class $[G' \rightarrowtail G \twoheadrightarrow G/G'] \in H^2(G/G', G')$, so that we have

Corollary 1. A metabelian group G is finitely presented if and only if the split extension $G' \rtimes (G/G')$ is finitely presented.

Another immediate consequence of Theorems A and B is the result stated in the title:

Corollary 2. If a soluble group G is finitely presented, so is every metabelian homomorphic image of G.

4. It is well known that the variety $\mathfrak{N}_2\mathfrak{U}$ of all nilpotent-of-class-2-by-Abelian groups contains finitely generated groups which are not residually finite and have infinitely generated centre. None of these pathologies can occur in the finitely presented case:

Corollary 3. If a soluble group G is finitely presented then every $\mathfrak{N}_2\mathfrak{U}$-quotient $\overline{G}$ of G is residually finite and satisfies max-n, the maximal condition for normal subgroups.

Proof. $\overline{G}$ contains a normal subgroup $N \lhd \overline{G}$ such that $Q = \overline{G}/N$ is Abelian and the centre Z of N contains the commutator subgroup N'. If G is finitely presented, so is $\overline{G}/Z$ by Corollary 2, hence Z is finitely generated as a Q-module and hence Noetherian. This shows that

$\overline{G}$ satisfies max-n. In order to prove that $\overline{G}$ is residually finite pick $1 \neq x \in \overline{G}$. The finitely generated Q-module Z is residually finite, hence Z contains a subgroup M of finite index which is normal in $\overline{G}$ and does not contain x. Replacing $\overline{G}$ by $\overline{G}/M$ we are reduced to the case where Z is finite. Then the centralizer of Z has finite index in $\overline{G}$ and we are reduced to the case where $\overline{G}$ is a finitely presented centre-by-metabelian group with finitely generated centre. In this situation it has been shown by J. R. J. Groves [4] that $\overline{G}$ is Abelian-by-polycyclic and hence $\overline{G}$ is residually finite by the result of J. Roseblade [7] and A. V. Jategaonkar [6].

5. Remarks. (1) H. Abels has pointed out (these proceedings, 205-11) that the multiplicative group G of all non-singular upper triangular matrice (a_{ij}) of rank 4 over $\mathbb{Z}[\frac{1}{2}]$ with $a_{11}=1=a_{44}$ is finitely presented, and thus supplied a surprisingly simple counter example to P. Hall's question. In fact G is in $\mathfrak{N}_3\mathfrak{U}$ and has centre $Z\cong\mathbb{Z}[\frac{1}{2}]$, so that the quotient G/Z is not finitely related. Moreover, quotients of G modulo cyclic central subgroups are non-Hopfian and hence are not residually finite. This shows that the statements of Corollaries 2 and 3 are sharp.

(2) A group G is termed almost finitely presented if there is a short exact sequence $R \rightarrowtail F \twoheadrightarrow G$ such that F is a free group of finite rank and R/R' is finitely generated as a G-module or, equivalently, if the trivial G-module $\mathbb{Z}$ admits a projective resolution which is finitely generated in dimensions 0, 1 and 2. It is plain that every finitely presented group is almost finitely presented, but whether the converse holds is still open. As in [3] one can show that Theorem A holds under the weaker assumption that G be almost finitely presented, whence we have

Corollary 4. Almost finitely presented metabelian groups are finitely presented.

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10 · Soluble groups with coherent group rings

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The objective of this note is to show how the key result of [4] which describes a structural property common to all (infinite) soluble groups of type $(FP)_2$ leads to a characterization of the soluble groups with coherent group ring.

1. Definition. We say that a group G is of type $(FP)_2$ over a commutative ring K with $1 \neq 0$, if the trivial KG-module K admits a KG-projective resolution $\underline{\underline{P}} \twoheadrightarrow K$ which is finitely generated in dimensions 0, 1, and 2; cf. [3], p. 20. A finitely presented group is of type $(FP)_2$ over every K, but little is known about the converse. Nevertheless the conceivably weaker condition of being of type $(FP)_2$ is often technically advantageous to work with, an instance of this phenomenon being provided by the sequel.

The key result of [4] is

Theorem A. *Let G be a soluble group and $N \lhd G$ a normal subgroup with infinite cyclic factor group G/N, and let $t \in G$ generate a complement of N in G. If G is of type $(FP)_2$ (over some commutative ring with $1 \neq 0$) then there is a finitely generated subgroup $B \leq N$ and a sign $\varepsilon = \pm 1$ such that $t^{-\varepsilon}Bt^{\varepsilon} \leq B$ and that G is the 'ascending' HNN-group*

$$G = \langle B,\ t;\ t^{-\varepsilon}bt^{\varepsilon} = \phi(b)\ \ (b \in B)\rangle,$$

where $\phi\colon B \rightarrowtail B$ is the restriction of conjugation by t.

This result will be used to establish the following

Proposition. Let G be a finitely generated soluble group with the property that every finitely generated subgroup is of type $(FP)_2$ over some commutative ring with $1 \neq 0$. Then G is an ascending HNN-group

$$G = \langle P,\ t;\ t^{-1}pt = \phi(p)\ \ (p \in P)\rangle$$

over a polycyclic base group P, or G is polycyclic.

2. The proof of this proposition will be divided into several steps. Recall a group is said to have finite rank (in the sense of Prüfer) if there is a natural number r such that every finitely generated subgroup can be generated by $\leq r$ elements (cf. [5], pp. 33-4). Let G be a group satisfying the assumptions of the proposition.

Claim. G has finite rank.

The proof goes by induction on the derived length of G starting at $G = \{1\}$. Since G is f. g. (short for: finitely generated) it has a subgroup H of finite index such that H contains the derived group G' of G and H/G' is free abelian of finite rank $n \geq 0$. Let $\{x_iG' \mid 1 \leq i \leq n\}$ be a basis of H/G' and put $N_i = gp(G', x_{i+1}, \ldots, x_n)$, $1 \leq i \leq n$. By Theorem A H is a HNN-group over a f. g. base group $B_1 \leq N_1$ and B_1 can be assumed to contain $x_2, x_3, \ldots, x_n$. By iteration one obtains in this way a tower of HNN-groups $H > B_1 > B_2 > \ldots > B_n$ with $B_i \leq N_i$ and $B_n \leq G'$. By the induction hypothesis B_n has finite rank. Since performing an ascending HNN-construction increases the rank by at most 1, we see that H and hence G have indeed finite rank. //

The fact that G is a f. g. soluble group of finite rank pins down the structure of G considerably: by a result of Mal'cev G contains a nilpotent normal subgroup $N \lhd G$ with $G/N = Q$ Abelian-by-finite (see

e. g. [5], Thm. 3.5, p. 79). In particular, $V = (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite dimensional vector space acted on by Q. For $q \in Q$ let χ_q be the characteristic polynomial of the action on V, let $\mathbb{Q}_m^0$ be the multiplicative group of the positive elements of $\mathbb{Q}$ and let $\Delta : Q \to \mathbb{Q}_m^0$ be the group homomorphism taking q to the absolute value of the determinant of the automorphism induced by q.

Claim. (i) For every $q \in Q$ either χ_q or $\chi_{q^{-1}}$ is integral.

(ii) The image of $\Delta : Q \to \mathbb{Q}_m^0$ is cyclic.

Let $b_1, \ldots, b_n$ be elements of N whose images in $V = (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$ form a basis, and let g be an element of $G-N$. Then $L = L_g = gp(b_1, \ldots, b_n, g)$ is a finitely generated nilpotent-by-cyclic group which, being a subgroup of G, is also of type $(FP)_2$. It follows from Theorem A (cf. [4], Thm. C) and g or g^{-1} induces an automorphism on $V_g = (gp_{L_g}(b_1, \ldots, b_n))_{ab} \otimes \mathbb{Q}$ with integral characteristic polynomial. This implies assertion (i), for the inclusion $gp_{L_g}(b_1, \ldots, b_n) \leq N$ induces by construction an isomorphism $V_g \tilde{\to} V$.

Because of (i) $\Delta(Q) \subseteq \mathbb{N} \cup \{1/k \mid k \in \mathbb{N}\}$. Since $\mathbb{Q}_m^0$ is free abelian this implies that $\Delta(Q) \leq \mathbb{Q}_m^0$ is either trivial or infinite cyclic. //

Claim. If $\Delta(Q) = \{1\}$ then G is polycyclic.

Pick elements $t_1, \ldots, t_n \in Q = G/N$ which freely generate a free Abelian subgroup T of finite index in Q. The characteristic polynomials $\chi_i = \chi_{t_i}$ are integral and have both leading and terminal coefficient ± 1, hence the $\mathbb{Z}gp(t_i)$-modules $A_i = \mathbb{Z}gp(t_i)/\chi_i . \mathbb{Z}gp(t_i)$ are f. g. as Abelian groups $(1 \leq i \leq n)$. Let $\bar{N}$ denote the quotient of N/N' modulo its torsion-subgroup. As χ_i annihilates $\bar{N}$ every cyclic T-submodule of $\bar{N}$ is a homomorphic image of $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ and hence f. g. as an Abelian group. Since $Q = G/N$ is finitely presented $\bar{N}$ is a f. g. Q-module, and also a f. g. T-module, and this proves that $\bar{N}$ is f. g. as Abelian group. On the other hand, $\mathbb{Z}Q$ is noetherian and so the torsion-subgroup of N/N' has bounded exponent and so, having also finite rank, must be finite. From the fact that N/N' is f. g. a well-known result

of Baer's permits us to conclude that N is polycyclic (see, e.g. [5], p. 55). Hence G is likewise polycyclic. //

If $\Delta(Q)$ is infinite cyclic pick an element $t_1 . N \in Q$ with $\Delta(Q) = \Delta(gp(t_1 . N))$. By Theorem A, with $\ker(G \twoheadrightarrow G/N \xrightarrow{\Delta} Q_m^0)$ resp. t_1 playing the rôle of N resp. t, G is an HNN-group over a f. g. base group $B \leq G$ and $\Delta(BN/N) = \{1\}$. N is the union of a tower

$$L < L_2 < L_3 < \ldots < N$$

of groups $L_j = (N \cap B)^{t_1^{\varepsilon j}}$. Since L_j and L_{j+1} are isomorphic and both have finite rank, L_j is of finite index in L_{j+1} (see e. g. [1], Lemma 10, p. 255). It follows that the inclusion $L = (N \cap B) < N$ induces an isomorphism $(L/L') \otimes \mathbb{Q} \tilde{\to} V = (N/N') \otimes \mathbb{Q}$. Thus the base group $B < G$ satisfies the conditions of the preceding claim with N replaced by L and Q replaced by B/L, and so G is an ascending HNN-group over a polycyclic base group, as asserted.

3. Definition. A ring Λ is termed right coherent if every finitely generated right ideal is finitely related. Similarly a group is termed coherent if every finitely generated subgroup is finitely related.

The preceding proposition leads to a characterization of finitely generated soluble groups having a right (or left) coherent integral group ring.

Theorem B. For a finitely generated soluble group G the following statements are equivalent:

(i) The group ring $\mathbb{Z}G$ is right coherent.

(ii) G is coherent.

(iii) G is an ascending HNN-group over a polycyclic base group or G is itself polycyclic.

Proof. (i) $\Rightarrow$ (iii). It is easy to see that if $\mathbb{Z}G$ is right coherent so is the group ring $\mathbb{Z}S$ for every subgroup $S \leq G$. If S is f. g. the augmentation ideal $I_S \lhd \mathbb{Z}S$ is f. g., hence S is of type $(FP)_2$, and the preceding proposition applies.

(ii)$\Longleftrightarrow$(iii). If (ii) holds the proposition gives (iii). Conversely it is straightforward to check that every f. g. subgroup G has the same structure as G. Since every polycyclic group and every ascending HNN-group with polycyclic base is finitely presented, G is coherent.

(iii)$\Rightarrow$ (i). As the group ring of a polycyclic group is noetherian and so a fortiori coherent, we can assume $G = \langle P, t; t^{-1}pt = \phi(p)\ (p \in P)\rangle$ with P polycyclic. To prove that $\Lambda = \mathbb{Z}G$ is right coherent it will do to show that $\mathrm{Tor}_1^{\Lambda}(\Pi\Lambda, M) = 0$ for every left Λ-module M and every direct product $\Pi\Lambda$ of copies of Λ (see, e. g. [6], p. 43). The Mayer-Vietoris sequence for HNN-groups [2] yields an exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}P}(\Pi\Lambda, M) \to \mathrm{Tor}_1^{\Lambda}(\Pi\Lambda, M) \xrightarrow{\partial} \Pi\Lambda \otimes_{\mathbb{Z}P} M \xrightarrow{\beta} \Pi\Lambda \otimes_{\mathbb{Z}P} M.$$

Since $\mathbb{Z}P$ is coherent and Λ is P-flat the left hand term is trivial. To see that β is injective consider the commutative square

$$\begin{array}{ccc} \Pi\Lambda \otimes_{\mathbb{Z}P} M & \xrightarrow{\ \beta\ } & \Pi\Lambda \otimes_{\mathbb{Z}P} M \\ \downarrow \mu & & \downarrow \mu \\ \Pi(\Lambda \otimes_{\mathbb{Z}P} M) & \xrightarrow{\ \Pi\beta'\ } & \Pi(\Lambda \otimes_{\mathbb{Z}P} M) \end{array}$$

where μ is given by $\mu(\Pi\lambda_i \otimes m) = \Pi(\lambda_i \otimes m)$, $m \in M$ and $\lambda_i \in \Lambda$. Since M is the union of f. g. $\mathbb{Z}P$-submodules and $\mathbb{Z}P$ is noetherian μ is injective. Moreover, each component $\beta' : \Lambda \otimes_{\mathbb{Z}P} M \to \Lambda \otimes_{\mathbb{Z}P} M$ is injective. So β itself is injective and $\mathrm{Tor}_1^{\Lambda}(\Pi\Lambda, M) = 0$. //

Postscript, January 1979. J. R. J. Groves, using similar methods, has also established the equivalence of statements (ii) and (iii) in Theorem B. His proof appeared in J. Australian Math. Soc. (A) 26 (1978), 115-25.

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11 · Cohomological aspects of 2-graphs. II

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This note contains a summary of a lecture given at the Symposium, together with some additions based on comments by Professor J.-P. Serre after the lecture. Some of the material appears elsewhere [1], in a different form.

Throughout, $\mathbb{F}$ denotes the field GF(2). An $\mathbb{F}$-vector space with given finite basis can be identified with the set of all subsets of the basis; addition corresponds to symmetric difference. In particular, a $(d-1)$-chain or $(d-1)$-cochain on an $(n-1)$-simplex is a collection of d-element subsets of a set X of n elements. A 1-chain or 1-cochain is thus a graph with vertex set X; 1-cycles are Eulerian graphs, and 1-cocycles are complete bipartite graphs. A two-graph is defined to be a 2-cocycle, that is, a set τ of triples with the property that any quadruple contains an even number of members of τ. Since 2-dimensional cohomology vanishes, τ is the coboundary of a 1-cochain k, that is, the set of triples carrying an odd number of edges of the graph k. The set of all graphs k giving rise to τ in this way is a switching class, a coset of the space C of 1-cocycles (complete bipartite graphs) in the space K of all graphs.

The annihilator of C in K (with respect to the duality between 1-chains and 1-cochains) is the space C^* of 1-cycles. Thus C^* is the dual space of K/C. Since this duality is invariant under the symmetric group on X, the numbers of isomorphism types of two-graphs and Eulerian graphs on X are equal. This was first proved by Mallows and Sloane by enumeration [3].

Given a set X of equiangular lines through the origin in Euclidean space $\mathbb{R}^d$, the set of triples of lines having the property that, for some choice of directions on the lines, all angles are acute, forms a two-graph. (For example, the diagonals of an icosahedron are an equiangular set. A triple of diagonals belongs to the two-graph if and only if it contains the three vertices of some face.) Conversely, any two-graph can be canoni-

cally represented in this way.

Much of the interest in two-graphs stems from the fact that many of the known doubly transitive finite permutation groups are automorphism groups of two-graphs. Among these are the groups $PSL_2(q)$ for $q \equiv 1 \pmod 4$, $PSU_3(q)$ for q odd, the Ree groups ${}^2G_2(q)$, $Sp_{2n}(2)$, the Higman-Sims group, and the Conway group $\cdot 3$. Indeed, it often happens that the largest number of equiangular lines in a given number of dimensions is associated with an interesting group (for example, 6 in 3 dimensions with the icosahedral group, 28 in 7 dimensions with $Sp_6(2)$, 276 in 23 dimensions with $\cdot 3$).

Let $\mathbb{F}X$ denote the vector space with basis X (the space of 0-cochains), and B the 1-dimensional subspace $\{\phi, X\}$. Then $\mathbb{F}X/B$ is the space C of 1-coboundaries; so there is an exact sequence

$$0 \to B \to \mathbb{F}X \to C \to 0 \qquad (*)$$

of $\mathbb{F}$-modules, where $B \cong \mathbb{F}$. (The image of a subset Y of X is the complete bipartite graph on Y and its complement.) These modules are $\mathbb{F}G$-modules for any group G of permutations of X.

Let τ be a two-graph on X, and G a group of automorphisms of τ. There are naturally-defined elements $\gamma \in H^1(G, C)$ and $\beta \in H^2(G, B)$, arising as follows. The operations of switching (adding elements of C) and the elements of G generate a group of permutations of the switching class of τ, which is a split extension CG. The stabilisers of members of the switching class form a conjugacy class of complements of C, corresponding to the cohomology class γ. (Thus $\gamma = 0$ if and only if G fixes a graph in the switching class.) Suppose τ is represented by a set of equiangular lines in Euclidean space $\mathbb{R}^d$. Then any element of G is represented by two orthogonal transformations of $\mathbb{R}^d$, one the negative of the other; so we have an extension $\hat{G}$ of a cyclic group of order 2 by G, corresponding to the cohomology class β. We have $\beta = 0$ if (*) splits as sequence of $\mathbb{F}G$-modules. Also, β is the image of γ under the connecting homomorphism associated with (*). For the icosahedron, we have $\gamma \neq 0$ (since the icosahedral group is doubly transitive on the set of diagonals and so leaves no non-trivial graph invariant), but $\beta = 0$

(since the full icosahedral group is $Z_2 \times A_5$). Mallows and Sloane [3] showed that $\gamma = 0$ for any cyclic group G. This was a tool in their enumeration of two-graphs mentioned earlier.

Now let G be a doubly transitive automorphism group of a non-trivial two-graph τ. Suppose that $\beta = 0$. Then the stabiliser G_x of an element of X (a line) has a subgroup $N(x)$ of index 2 (the stabiliser of a direction on the line). It is clear geometrically that $N(x) \cap G_y \subseteq N(y)$ for any x and y; that is, $N(x)$ is strongly closed in G_x. Alternatively, $N(x)$ is the kernel of a non-zero homomorphism from G_x to $\mathbb{F}$, and so corresponds to an element $\alpha_x \in H^1(G_x, \mathbb{F})$. The strong closure condition asserts that, for any x and y, the restrictions of α_x and α_y to $G_x \cap G_y$ are equal; call α_x strongly closed in this case.

Since G is doubly transitive, $\gamma \neq 0$. The exactness of

$$\ldots \to H^1(G, \mathbb{F}X) \to H^1(G, C) \to H^2(G, B) \to \ldots$$

and the vanishing of β ensures that γ is the image of a nonzero element $\gamma^* \in H^1(G, \mathbb{F}X)$. Note that $H^1(G, \mathbb{F}X)$ and $H^1(G_x, \mathbb{F})$ are isomorphic by Shapiro's lemma; γ^* and α_x correspond under this isomorphism.

Of course, a strongly closed element of $H^1(G_x, \mathbb{F})$ could also arise as the restriction of an element of $H^1(G, \mathbb{F})$. Theorems asserting that, under certain hypotheses, an element of $H^1(G_x, \mathbb{F})$ is such a restriction, are transfer theorems. Such a theorem is the following result [2], [4], the converse of our previous observations.

Theorem. Let G be a finite doubly transitive permutation group on a set X. Suppose that, for $x \in X$, G_x has a strongly closed subgroup $N(x)$ of index 2. Then either

(i) G has a subgroup N of index 2 with $N \cap G_x = N(x)$;

or (ii) G is an automorphism group of a nontrivial two-graph with $\beta = 0$.

A more general result can be proved, in which the condition $\beta = 0$ does not appear. If $\beta \neq 0$, then the distinguished subgroup $N(x)$ of index 2 in G_x no longer exists; in the projection from $\hat{G}$ onto G, the stabiliser of a direction is mapped onto G_x. It is tempting to say that

$\alpha_x = 0$ in this case, but this may not make sense. Note that the restriction of β to G_x is always zero.

There are possible generalisations, to infinite sets, non-faithful group actions, 'oriented two-graphs', fields other than GF(2), or higher dimensional cocycles (see [1]). Little is known about most of these.

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12 · Recognizing free factors

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1. Introduction

The augmentation ideal I_G of a group G is the kernel of the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ where $\varepsilon(\Sigma n_g g) = \Sigma n_g$. If $H \leq G$, $I_H G$ denotes the right ideal of $\mathbb{Z}G$ generated by I_H. It was shown by Jacques Lewin [3] that $G = K *_H L$ if and only if $I_G = I_K G + I_L G$ where $I_K G \cap I_L G = I_H G$. In particular $G = K * L$ if and only if $I_G = I_K G \oplus I_L G$ (see also [1, Theorem 4. 7]).

In general it is not true that if $I_H G$ is a direct summand of I_G then H is a free factor of G. In particular if G is finite, then $I_H G$ is a direct summand of I_G if and only if G is a Frobenius group with complement H [4, p. 59]. However D. E. Cohen [1] (following the work of Swan [6]) showed that if G is a finitely generated torsion free group and $I_H G$ is a direct summand of I_G, then H is a free factor of G. In this paper, partial results are obtained for the case when it is only assumed that H is torsion free. In [2] I showed that H is a free factor of G if $I_G / I_H G$ is a finitely generated projective $\mathbb{Z}G$-module.

As in [2] a pair (G, H) is defined to be a group G and a subgroup H of G. The pair (G, H) is said to be finitely generated if G is generated by $H \cup S$ where S is finite. Again as in [2], a finitely generated pair (G, H) is said to be accessible if G can be regarded as the fundamental group $\pi(\mathcal{G}, X)$ of a graph of groups $(\mathcal{G}, X)$ satisfying the following conditions.

(i) The underlying graph X is finite.

(ii) Every edge group is finite.

(iii) For some vertex P_0, $H \leq G_{P_0}$ and

Res: $H^1(G_{P_0}, \mathbb{Z}G_{P_0}) \to H^1(H, \mathbb{Z}G_{P_0})$ is injective.

(iv) If $P \neq P_0$, then $H^1(G_P, \mathbb{Z}G_P) = 0$, i. e. G_P has at most one

end.

It is proved in [2] that a finitely generated pair (G, H) is accessible if and only if K(G, H), the kernel of the restriction mapping $H^1(G, \mathbb{Z}G) \to H^1(H, \mathbb{Z}G)$, is a finitely generated $\mathbb{Z}G$-module. In fact the result is proved with a general commutative ring R replacing $\mathbb{Z}$. In this paper attention is restricted to the case $R = \mathbb{Z}$.

A finitely generated group G is said to be accessible if (G, 1) is an accessible pair. It follows from [2, Theorem 5.8] that if G is accessible, then (G, H) is an accessible pair for every subgroup H.

Theorem 1. Let H be a torsion-free subgroup of the group G, and suppose (G, H) is an accessible pair. If $I_H G$ is a direct summand of I_G, then H is a free factor of G.

Theorem 2. Let H be an infinite cyclic subgroup of the finitely generated group G. If $I_H G$ is a direct summand of I_G, then H is a free factor of G.

The proof of Theorems 1 and 2 which are given in §2 and §3 depend heavily on the results and techniques of [2].

If H is infinite cyclic then $I_H \cong \mathbb{Z}H$. If $H \leq G$, then $I_H G \cong \mathbb{Z}G$. It follows from [2, Lemma 5.1] that Res: $H^1(G, \mathbb{Z}G) \to H^1(H, \mathbb{Z}G)$ is surjective if and only if Hom $(I_G, \mathbb{Z}G) \to$ Hom $(I_H G, \mathbb{Z}G)$ is surjective. But $\mathrm{Hom}(I_G, I_H G) \to \mathrm{Hom}(I_H G, I_H G)$ is surjective if and only if $I_H G$ is a direct summand of I_G. Thus, by Theorem 2, Res: $H^1(G, \mathbb{Z}G) \to H^1(H, \mathbb{Z}G)$ is surjective if and only if H is a free factor of G. This is precisely the result that is needed to strengthen the unknotting criterion of Swarup [7].

Let $f : S^n \to S^{n+2}$ be a locally flat PL embedding of the n-sphere in the (n+2)-sphere. Let T be a regular neighbourhood of $f(S^n)$ in S^{n+2} and let M be the closure of $S^{n+2} - T$. Let S be a copy of $f(S^n)$ in ∂T which bounds an (n+1)-submanifold of M. In [7] G. A. Swarup shows that if S is homotopically trivial in M and if $\pi_1(M)$ is accessible, then M has the homotopy type of a circle. It follows that if $n = 1$ or $n \geq 3$, then f is unknotted. The only place in Swarup's proof where accessibility is used is in proving that if H is an infinite cyclic subgroup

of an accessible group G and $\mathrm{Res}: H^1(G, \mathbb{Z}G) \to H^1(H, \mathbb{Z}G)$ is surjective, then H is a free factor of G. As has been remarked above, the assumption that G is accessible can be replaced by the weaker requirement that G be finitely generated. Thus we have the following result.

Unknotting criterion. If S is homotopically trivial in M, then M has the homotopy type of a circle.

For $n = 1$, this result is well-known from [5]. In a similar way it is possible to remove the accessibility condition from Theorem 3.6 of [8].

2. Proof of Theorem 1

Let H be a non-trivial torsion-free subgroup of the group G and suppose (G, H) is an accessible pair. Let $(\mathcal{G}, X)$ be a graph of groups satisfying conditions (i)-(iv) of §1. If $I_H G$ is a direct summand of I_G, then $I_H L$ is a direct summand of I_L for any subgroup L, $H \leq L$, by [1, Lemma 4.4]. Let $L = G_{P_0}$, the vertex group of $(\mathcal{G}, X)$ containing H. Now $K(L, H) = 0$, and so it follows from [2, Lemma 5.1] that $\mathrm{Hom}(I_L, \mathbb{Z}L) \to \mathrm{Hom}(I_H L, \mathbb{Z}L)$ is injective. Since $I_H L$ is a direct summand of I_L and $I_L \subset \mathbb{Z}L$, it follows that $H = L$. Let $e \in E(X)$ be such that $o(e) = P_0$. Since H is torsion free, G_e is trivial. It follows immediately that H is a free factor of G.

3. Proof of Theorem 2

Let $M = G *_H L$ where G, L are finitely generated groups and H is infinite. In the associated Mayer-Vietoris sequence [4]

$$H^0(H, \mathbb{Z}M) \to H^1(M, \mathbb{Z}M) \to H^1(G, \mathbb{Z}M) \oplus H^1(L, \mathbb{Z}M) \to H^1(H, \mathbb{Z}M)$$

$H^0(H, \mathbb{Z}M) = 0$, since H is infinite and $H^1(G, \mathbb{Z}M) \cong H^1(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}M$. Now the kernel of the restriction mapping $H^1(G, \mathbb{Z}M) \to H^1(H, \mathbb{Z}M)$ can be regarded as $K(G, H) \otimes_{\mathbb{Z}G} \mathbb{Z}M$ and there is an isomorphic preimage in $H^1(M, \mathbb{Z}M)$.

Lemma 1. Let G be a finitely generated group and let S be a finite subset of $H^1(G, \mathbb{Z}G)$. Then G can be regarded as the fundamental group $\pi(\mathcal{G}, X)$ of a graph of groups $(\mathcal{G}, X)$ satisfying the following conditions.

(i) The underlying graph X is finite.

(ii) Edge groups are finite.

(iii) For each vertex P, $S \subset K(G, G_P)$.

A proof of Lemma 1 is not included as it is essentially a repeat of the argument of [2, Theorem 5.5]. The argument given there is specifically for the case when S is a finite generating set for K(G, H) where H is a subgroup of G, but there is no difficulty in adapting the argument to the case considered here.

Suppose now that H is an infinite cyclic subgroup of G and $I_H G$ is a direct summand of I_G. Thus Res: $H^1(G, \mathbb{Z}G) \to H^1(H, \mathbb{Z}G)$ is surjective. Now H has two ends and so $H^1(H, \mathbb{Z}H) \cong \mathbb{Z}$. It follows that $H^1(H, \mathbb{Z}G) \cong \mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is generated as a $\mathbb{Z}G$-module by a single element s. Let $S = \{\sigma\}$ where $\mathrm{Res}(\sigma) = s$. Let $G = \pi(\mathcal{G}, X)$ be the corresponding decomposition of G as given in Lemma 1. Suppose that for some vertex P, G_P has more than one end. Choose a finitely generated group L_P containing G_P such that L_P has one end, e.g. we could take $L_P = G_P \times F$ where F is free abelian of rank two. Form $G_1 = G *_{G_P} L_P$. Repeating this process a finite number of times one eventually obtains an accessible group A. Now $\sigma \in K(G, G_P)$ for every vertex P. By the remarks at the beginning of the section it follows that there exists $\sigma_1 \in H^1(A, \mathbb{Z}A)$ such that $\mathrm{Res}(\sigma_1)$ generates $H^1(H, \mathbb{Z}A)$. Thus Res: $H^1(A, \mathbb{Z}A) \to H^1(H, \mathbb{Z}A)$ is surjective. It has been seen that this implies that $I_H A$ is a direct summand of I_A. Therefore by Theorem 1, H is a free factor of A. By the Kurosh Subgroup Theorem, H is a free factor of G. This completes the proof of Theorem 2.

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13 · Trees of homotopy types of (π, m)-complexes

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Let π be a group and m an integer ≥ 2. A (π, m)-complex X is a finite, connected CW complex with dimension $\leq m$ having fundamental group $\pi_1 X$ isomorphic to π and trivial homotopy modules $\pi_i X$ for $1 < i < m$. Examples of (π, m)-complexes are finite, connected two-complexes $(m = 2)$, lens spaces, and the m-skeleton of any Eilenberg-MacLane space of finite (cell) type. Also, if X is a (π, m)-complex, then so is $X \vee iS^m$, where iS^m is a bouquet of i spheres of dimension m.

Problem. For a given (π, m), classify the homotopy types of such complexes.

This homotopy classification can be conveniently expressed using the language of trees. The homotopy tree $HT(\pi, m)$ is the directed tree whose vertices consist of the homotopy types of (π, m)-complexes; vertex $[X]$ is joined by an edge to vertex $[Y]$ iff $X \vee S^m \simeq Y$. The tree is connected by a theorem of J. H. C. Whitehead [Wh, theorem 14] which says that for any two (π, m)-complexes X, Y there are integers i, j such that $X \vee iS^m \simeq_s Y \vee jS^m$. There are clearly no circuits. We define the minimal Euler characteristic $\chi_{min}(\pi, m) = \min\{(-1)^m \chi(X) \mid X \text{ is a } (\pi, m)\text{-complex}\}$. The level $\ell(X) = (-1)^m \chi(X) - \chi_{min}(\pi, m)$. X is a root if X has no predecessor in the tree; a minimal root (minimal complex) if $\ell(X) = 0$. The problem now is to describe the tree $HT(\pi, m)$.

1. Finite fundamental group

The form of the trees $HT(\pi, m)$ for π a finite group is taking definite shape. We say that a finitely generated, torsion free π-module (lattice) M satisfies the Eichler condition (E) if the semi-simple Q-algebra $\mathrm{End}_{Q\pi}(QM)$ (Q is the rationals) has no simple component which

is a totally definite quaternion algebra over its center [S_1, page 176].

Theorem. Let π be a finite group. Any two (π, m)-complexes at level $\ell \geq 2$ have the same homotopy type [Wi], [D_5]. If X is a minimal (π, m)-complex and $\pi_m X \oplus Z\pi$ has (E), then any two complexes at level 1 have the same homotopy type [Br].

It is remarkable that $\pi_m X \oplus Z\pi$ nearly always satisfies (E). For example, for π finite, $\chi_{min}(\pi, m) \geq 0$ [S_2]; by [D_1, prop. 5.1] $\pi_m X \oplus Z\pi$ has (E) provided $\chi_{min}(\pi, m) > 0$. This always holds if m is even.

Proposition. Let π be a finite group. If $\chi_{min}(\pi, m) = 0$, then π must be periodic of period $m + 1$. Furthermore, the minimal (π, m)-module in this case is the trivial π-module Z.

Proof. Let X be a (π, m)-complex such that $\chi(X) = 0$. Arguing on the cellular chain complex of the universal cover $\tilde{X}$ of X, one shows that Z-rank $\pi_m X = 1$. Hence, $\tilde{X} \simeq S^m$ and π acts on $\tilde{X}$ without fixed points. This forces m to be odd and π to act via orientation preserving homeomorphisms [H, p. 290]. Thus $Z = \pi_m X$ is the trivial π-module and π is periodic with period $m + 1$. //

Furthermore, $Z \oplus Z\pi$ has (E) if $Z\pi$ has (E). Now let $\mathcal{P}$ be the class of all pairs (π, m) such that there is a single homotopy type in $HT(\pi, m)$ at each level $\ell > 0$. We see that the following pairs are in $\mathcal{P}$: (a) π not periodic and $m \geq 2$, (b) π is periodic, $Z\pi$ satisfies (E), and $m \geq 2$, (c) π is periodic, and $m + 1$ is not a period of π. Then the homotopy tree looks at worst like figure 1A:

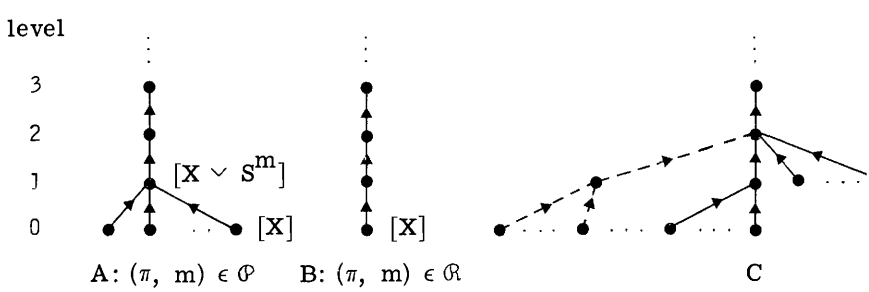

Figure 1. $HT(\pi, m)$ for π-finite.

Let $\mathcal{R} \subset \mathcal{P}$ consist of pairs (π, m) such that $HT(\pi, m)$ has a single minimal root. Then $\mathcal{R}$ contains (π, m) such that (a) $\pi = Z_n$ and m even [DSI], [CS], [D_1], (b) $\pi = Z_m \times Z_n$ and m even [D_2], and (c) $\pi = D_{2n}$, the dihedral group of order $2n$ (n odd), and m even. See figure 1B.

The worst possible tree is given by figure 1C. Such non-minimal roots exist [D_4]; the dotted lines indicate a question. The Jordan-Zassenhaus theorem [S_1] shows that there are only finitely many vertices at each level. For π finite abelian, distinct minimal roots were discovered by W. Metzler [M] and a lower bound on their number is given in [Si] and [SD].

2. Infinite fundamental group

For π infinite, very little is known. If π admits a $K(\pi, 1)$ which is a (π, m)-complex, then there is a single minimal root in $HT(\pi, i)$ for $i \geq m$ [C]. Furthermore, in this case, the tree $HT(\pi, i)$ $(i \geq \max(m, 3))$ is identical with the tree of isomorphism classes of stably free, finitely generated projective π-modules (what about $\pi = Z^n$?). If π is finitely generated and free, then $HT(\pi, i)$ looks like figure 1B [Wa], [B]. An intriguing example of M. Dunwoody [Du] exhibits a non-minimal root at level 1 in $HT(T, 2)$, where T is the trefoil group. For π a finitely generated abelian group of rank n with torsion and $m < n$, there are distinct minimal roots [D_3].

Problems. (a)* For π finite abelian, how many distinct homotopy types are there at level 0?

(b) Add to the list of pairs $(\pi, m) \in \mathcal{R}$.

(c) Do there exist (π, m)-complexes X, Y such that $X \vee 2S^m \simeq Y \vee 2S^m$ but $X \vee S^m \not\simeq Y \vee S^m$?

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* Note added in proof: Wes Browning has shown (preprint, ETH, Zurich) that the minimal roots are just those found by Metzler [M].

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14 · Geometric structure of surface mapping class groups

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§1. Introduction

This article will describe some recent progress on the structure of the group Γ_g of mapping classes for a compact surface S_g of genus $g \geq 2$. A mapping-class is an isotopy class of homeomorphisms (usually assumed to be C^∞ diffeomorphisms); occasionally it will be convenient to use the alternative definition of it, valid by virtue of Nielsen's theorem, as an element of the outer automorphism group of the fundamental group $\pi_1(S_g)$.

I shall not attempt here to catalogue the many ways in which these groups impinge on various parts of mathematics, nor will their properties be developed comprehensively. My concern is with two aspects of the theory, which bear a close relationship to each other. One of them is the purely combinatorial study of how Γ_g operates on the space of simple loops in S_g, and the other is the geometric action as the Teichmüller modular group on the classifying space $T_g = T(S_g)$ of complex structures on the surface S_g. It transpires that in attempting to analyse the boundary structure of $T(S_g)$ and the extended action of Γ_g on it, one is naturally led to the former question.

My primary aim in the description of Teichmüller space (§§ 3, 4) which forms the basis of this account has been to provide sufficient background to understand the geometric formulation of Thurston's recent theorem on classification of mapping-classes, in terms of both their action on $T(S_g)$ and the dynamical systems determined by them on S_g. The last two sections describe some of my own related work on the boundary action of Γ_g and certain group theoretical properties of Γ_g which derive from the earlier discussion.

It should be noted that part of §4 appears in more extended form in the paper [4] of Bers, which gives a proof of the classification theorem

independent of Thurston's results. In particular the interpretation of Teichmüller's theorem in terms of measured foliations comes from there.

I should like to acknowledge here my gratitude to the many people who have contributed to my understanding of the material presented; in particular I have benefited greatly from discussions with Bill Abikoff, Lipman Bers and John Hubbard.

§2. Geometric classification of diffeomorphisms

If S is a surface of genus 1, then any diffeomorphism of S may be viewed as a self-mapping of the plane preserving the lattice of integer points, and the mapping classes in Γ_1 correspond to elements of $SL_2(\mathbb{Z})$. Note that here and in the sequel we restrict attention to orientation-preserving maps.

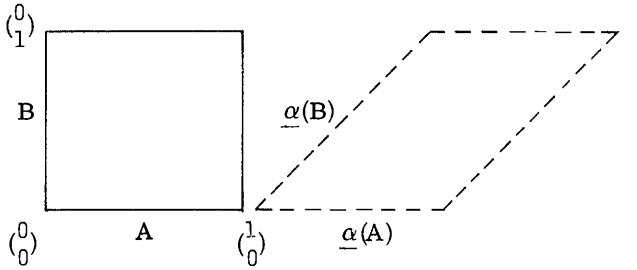
The algebraic classification of the matrices has two geometrical interpretations, one in terms of the action on the plane (and on S) and the other stemming from the action as fractional linear transformations of the upper half plane U. We review them briefly now as a significant first impression of the general pattern.

Definition. A matrix in $SL_2(\mathbb{Z})$ is termed <u>elliptic</u>, <u>parabolic</u> or <u>hyperbolic</u> according as the value of $(\text{Trace})^2$ is < 4, $= 4$ or > 4.

Elliptic matrices are conjugate to rotations and constitute the torsion part of Γ_1, forming two conjugacy classes of maximal cyclic subgroups of orders 4 and 6. Such elements act on U as non-Euclidean rotations fixing some interior point $\tau \in U$; this occurs precisely when the lattice Λ_τ generated by 1 and τ admits a complex multiplication, which corresponds to the fact that the Riemann surface $\mathbb{C}/\Lambda_\tau$ has a conformal automorphism distinct from the canonical involution.

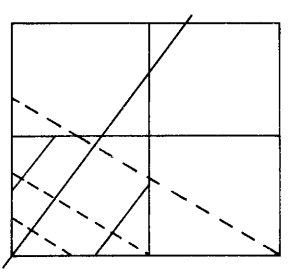
Parabolic matrices are typified by the element $\underline{\alpha} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Each fixes a single boundary point of U in the Γ_1-orbit $\mathbb{Q} \cup \{\infty\}$. In the plane, $\underline{\alpha}$ fixes the foliation by horizontal lines, which projects to an invariant foliation on the torus with closed leaves. Notice that the loops in S transverse to the fixed 'horizontal' loop are subjected to a 'shear' map, repre-

sented by the dotted lines in the diagram below.



Modulo isotopy, this is equivalent to the result of cutting along the horizontal loop A, applying a 360° twist and rejoining the edges, a manoevre known as a Dehn twist about A.

In contrast, hyperbolic elements have distinct eigenvalues λ and λ^{-1}, with $\lambda > 1$ an algebraic integer. There are then two irrational fixed points in $\mathbb{R}$, and the action on U is a non-Euclidean translation from one point in the direction of the other one. By its action on the torus a hyperbolic element generates an Anosov-flow: it fixes two mutually transverse foliations corresponding to the λ and λ^{-1} eigenvectors. There are no closed leaves in this case. One visualises the diffeomor-



Segments of two transverse leaves on the torus.

phism as a map which preserves the two foliations while expanding distance by a factor λ along the one (called unstable) and contracting by λ^{-1} along the other (stable) foliation. For further discussion, consult [1, 10].

When the genus is at least 2, there is no immediate classification of diffeomorphisms, except for torsion. Nielsen [9] proved that a periodic mapping-class must contain a homeomorphism of the same order, and again it can be shown that this determines a complex structure on the surface such that the mapping is a conformal automorphism.

The rest of the picture has come into focus only recently, following Thurston's work on singular foliations of surfaces (reported briefly in [11]).

Theorem. Each mapping-class of infinite order in Γ_g ($g \geq 2$) contains a diffeomorphism of precisely one of the following types:

(i) reducible: leaves invariant a non-trivial set of loops in S;

(ii) pseudo-Anosov: fixes two mutually transverse singular foliations.

Note. A reducible diffeomorphism may permute loops rather than preserve each one. It represents a mapping of the surface of lower genus obtained by cutting up S along the various loops (of course this may not be connected). Examples of this type are readily constructed using Dehn twists and permutations. They are of course the analogue of parabolic matrices in Γ_1. We shall elaborate on the second category of diffeomorphism in §4 after some preliminary discussion of the higher-genus analogue of U.

§3. Teichmüller space and the modular group

In order to simplify the exposition, here and elsewhere we are dealing only with compact surfaces S_g, although the entire theory extends to the case of surfaces with finite boundary $S_{g,n}$, and a comprehensive treatment even for the case in hand would involve the more general setting as we shall see in §5.

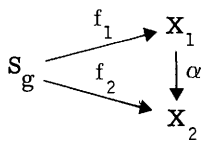
There are two formulations for the base space of the geometric action of Γ_g, one in terms of S_g and the other involving the group $G = \pi_1(S_g)$ regarded as a Fuchsian group. It is convenient to use both.

Definition. A marked complex structure on S_g is a diffeomorphism $f: S_g \to X$ with X a Riemann surface.

A marked complex structure determines an isomorphism $r = r_f$ from G onto a co-compact Fuchsian group $G_X \subseteq L = \mathrm{Aut}\, U$, with $X \cong U/G_X$, by the classical uniformisation theorem. Notice that (r_f, G_X) is unique up to composition of r_f with an automorphism of U, which

changes G_X to a conjugate group in L.

The Teichmüller space T_g of S_g is a classifying space for complex structures on S_g, in the sense that its points are equivalence classes of marked structures. Here two structures (f_1, X_1), (f_2, X_2) are regarded as equivalent if there is a morphism $\alpha: X_1 \to X_2$ such that the diagram commutes up to isotopy. An equivalence class of structures on



S_g is uniquely specified by a set of isomorphisms $\{\gamma \circ r, \gamma \in L\}$ having discrete co-compact images in L. Notice the analogy with the genus 1 case, where a class of complex structures is specified by a choice of generators for a lattice subgroup $\Lambda \subseteq \mathbb{C}$, up to multiplication of each by a non-zero complex number.

One useful way to construct a parametrisation of T_g will be described now - we shall need to describe a different one in §4 in order to inter-relate the type of a diffeomorphism and the geometric action on T_g. Choose a partition of S, that is, a maximal collection of simple loops in S_g which are disjoint and which define mutually distinct non-trivial isotopy classes. It is a matter of elementary surface topology to show that they are $3g - 3$ in number and divide S_g into a union of three-holed spheres ('parts'). To specify a class of structures on S_g we measure the traces of a set of matrices in $SL_2(\mathbb{R})$ determined thus:- fix a choice of decomposition for G into a collection of subgroups which represent the various component parts, with appropriate amalgamations over the various cyclic infinite subgroups representing the loops; if $r: G \to L$ is a representative homomorphism for the class of complex structures, there is a set of $6g - 6$ matrices representing elements of L whose traces determine the class of r. From these, $3g - 3$ give lengths of geodesic loops, while the others provide a measure of how the two banks of a loop are fitted together in assembling the surface from component parts. More details of this rather intricate procedure may be found in [14, chap. 9] and references cited there.

Using roughly the same description of T_g, Fricke gave in essence a proof of the discontinuity of the group Γ_g nearly 50 years ago. The action of Γ_g on T_g is defined as follows. Let $[f]$, $[\alpha]$ denote classes of structure $f: S \to X$ and diffeomorphism α. Then the rule $\rho: \Gamma_g \times T_g \to T_g$ obtained by setting

$$([\alpha], [f]) \overset{\rho}{\mapsto} [f \circ \alpha^{-1}]$$

determines a Γ_g-action as real analytic homeomorphisms of T_g. Equivalently one can let Γ_g operate on classes of homeomorphisms by

$$([\alpha], [r_f]) \mapsto [r_{f \circ \alpha^{-1}}] = [r_f \circ \hat{\alpha}^{-1}],$$

with $\hat{\alpha} \in \mathrm{Aut}(\pi_1(S))$ induced by $\alpha: S \to S$.

Theorem. The ρ-action of Γ_g is properly discontinuous.

We sketch the proof. Let $\{f_n: S \to X_n, n = 1, 2, \ldots\}$ be a sequence of structures which converges to $f: S \to X$, with $\{r_n: G \to G_n\}$ a corresponding sequence of isomorphisms tending to $r: G \to G'$. Let $\beta_n = r_n \circ r^{-1}: G' \to G_n$: since the $\{r_n\}$ are all Γ_g-equivalent, we may assume that the β_n are automorphisms of G'. Each β_n permutes the set of traces of group elements, which has no finite accumulation point, so almost all β_n actually fix all traces in G'. Therefore they extend to automorphisms of the Lie group L, since G' contains non-commuting hyperbolic elements. This implies that the sequence $\{[f_n]\}$ of classes terminates after finitely many steps, so no Γ_g-orbit accumulates in T_g.

Notes. 1. The action is effective if $g \geq 3$. In Γ_2, as in Γ_1, there is a central involution which fixes the whole space; this corresponds to the automorphism of interchanging sheets which all surfaces of genus 1 and 2 possess.

2. T_g carries a structure of complex manifold, in which the action of Γ_g is biholomorphic. Remarkably, Γ_g is the full group of automorphisms of T_g in this structure, by a theorem of Royden. The quotient space by the action is the moduli space of Riemann surfaces $\mathfrak{X}_g$, each point of it representing a Riemann surface modulo biholomorphic

equivalence; the discontinuity of Γ_g implies that $\mathfrak{X}_g$ inherits a natural structure of complex V-manifold from T_g.

3. Since T_g is a topological cell, it follows by the Smith fixed point theorem that all torsion elements of Γ_g fix a non-empty set; this implies the result of Nielsen mentioned in §2. A closer analysis shows that the fixed set of a mapping class $[\alpha]$ is a complex submanifold of T_g isomorphic to a Teichmüller space for the ramified quotient surface $S/\langle\alpha\rangle$. Further consequences are a precise upper bound of $4g + 2$ for the order of torsion elements in Γ_g (due originally to Wiman), and the finitude of the number of conjugacy classes of torsion subgroups ([7]).

§4. Quadratic differentials and measured foliations

The connexion between type of a diffeomorphism (§2) and the geometry of its action on T_g comes from a deep theorem of Teichmüller which relates the structure of T_g to quadratic differentials on a reference surface. For a detailed account the reader should refer to [2, 8]; we present only an outline sufficient to exhibit the dichotomy of Thurston's theorem.

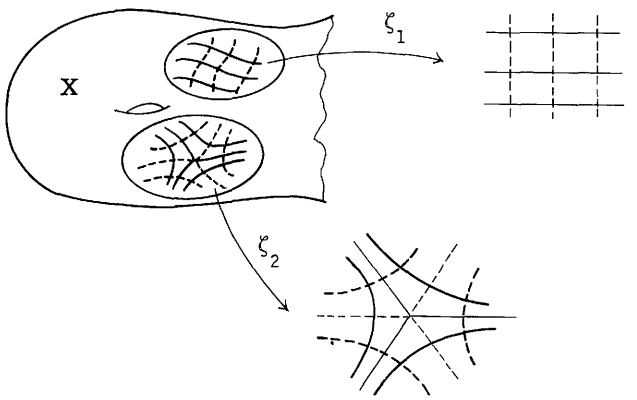
Let $f: S_g \to X$ be a fixed complex structure. We recall that a holomorphic quadratic differential Φ on X is a global section of the second power of the canonical (cotangent) bundle of X; this is in other words a collection of holomorphic functions $\phi(z)$ in the local coordinates z on X, which transform under change of parameter in such a way that $\phi(z)dz^2$ remains invariant. The vector space $Q(X)$ of all such Φ on X has (real) dimension $6g - 6$ by the Riemann-Roch theorem.

Each $\Phi \in Q(X)$ determines a canonical pair of foliations on X, with algebraic singularities at the zeros of Φ; their leaves are the horizontal and vertical trajectories of Φ, that is, the curves in X which are mapped into horizontal and vertical lines in $\mathbb{C}$ under the local mappings

$$z \mapsto \zeta(z) = \int_{z_0}^{z} \phi(z)^{\frac{1}{2}} \cdot dz \, .$$

We shall refer to them as the horizontal and vertical foliations of ϕ. Of course they could equally well be defined as the foliations associated to the closed real forms $\operatorname{Im} \Phi^{\frac{1}{2}}$, $\operatorname{Re} \Phi^{\frac{1}{2}}$ on X. Note that there is a natural

way to measure distance between leaves, using the Riemannian metric $ds = |d\zeta|$. Near a zero of Φ there are $n + 2$ distinct branches of ζ^{-1}, where n is the order of the zero. The picture below shows a simple zero and a generic point.



We are ready to state the beautiful theorem of Teichmüller which places any pair of marked complex structures f_1, f_2 in a precise geometrical relationship.

Theorem. *There exist quadratic differentials Φ_1, Φ_2 on the Riemann surfaces X_1, X_2 and a diffeomorphism $\omega: X_1 \to X_2$ isotopic to $f_2 \circ f_1^{-1}$ such that in the local coordinates ζ_j given by Φ_j $(j = 1, 2)$, ω is given by*

$$\zeta_1 \mapsto \zeta_2 = K^{\frac{1}{2}} \cdot \mathrm{Re}\ \zeta_1 + iK^{-\frac{1}{2}}\ \mathrm{Im}\ \zeta_1 ,$$

where $K = K(\omega)$ is constant on X_1.

Note. 1. The geometric interpretation is that ω carries the horizontal and vertical foliations of Φ_1 into those of Φ_2, multiplying distance along horizontal leaves by $K^{-\frac{1}{2}}$ and along vertical leaves by $K^{\frac{1}{2}}$. This amounts to a K-quasiconformal diffeomorphism with constant 'dilatation', i. e. the tangent mapping to ω at a point of X_1 distorts by a constant factor K everywhere on X_1 (except at zeros of Φ_1) taking tangent unit circles into ellipses with axis-ratio K. If $K = 1$, then ω is

biholomorphic, and the two structures f_1, f_2 are equivalent.

One should regard the map ω as the solution to an extremal problem, that of minimising inside a given mapping-class the norm of the distortion function over the whole surface. The extremal map is unique, in that any other diffeomorphism distorts by a factor greater than K on a non-negligible set; the quadratic differentials are unique up to a real multiplicative constant.

Note. 2. An immediate consequence of the theorem is the global metric d on T_g determined by

$$d([f_1], [f_2]) = \log K(\omega).$$

The final ingredient for the representation of Γ_g is provided by the observation that the cotangent space to T_g at the point given by $[f]$ with $f: S \to X$ is canonically isomorphic with $Q(X)$. This corresponds intuitively to the fact that every complex structure near $[f]$ may be achieved, starting from $[f]$, by specifying some differential Φ and applying an extremal 'Teichmüller' mapping of small distortion $K > 1$. Such a mapping can be characterised as the solution $\omega = \omega(k, \Phi)$ to a certain first order partial differential equation (Beltrami's equation),

$$\frac{\partial \omega}{\partial \bar{z}} = k \frac{\bar{\phi}(z)}{|\phi(z)|} \cdot \frac{\partial \omega}{\partial z}$$

in local coordinates on X, with $k = \frac{K - 1}{K + 1}$, $0 \leq k < 1$. The homeomorphism $\omega(k, \Phi)$ amounts to following the flow determined by the horizontal Φ-foliation for time $K^{-\frac{1}{2}}$ and the vertical flow for time $K^{\frac{1}{2}}$.

From this local picture we can define a covering of T_g by rays in direction Φ emanating from $[f]$, noting that $\omega(k, \Phi) = \omega(k, \lambda\Phi)$ for any real $\lambda > 0$. The end result is a representation of T_g as a unit ball in $\mathbb{R}^N$, $N = 6g - 6$, with the added bonus that there is a natural way to extend it to a closed ball by associating to the ray $\{\omega(k, \Phi), 0 \leq k < 1\}$ the end-point $\Omega(\Phi) = \lim_{k \to 1-} \omega(k, \Phi)$, which is viewed as the (stable) horizontal Φ-foliation of X. After checking that this process is independent of the choice of base point $[f]$, we have the basis for a complete description of the classification.

Proposition. *There is a compactification of* T_g *as a closed unit ball, whose boundary points consist of measured foliations on* S_g, *on which* Γ_g *acts as a group of diffeomorphisms.*

The precise comparison with the boundary constructed by Thurston will not concern us here. We observe that the action of Γ_g on the boundary sphere is defined by a sort of co-adjoint representation to the action $\rho: \Gamma_g \times T_g \to T_g$: conjugation by α in $\mathrm{Diff}_1(S)$[†] differentiated at the identity (e) induces a linear map on the tangent space at (e) and a similar procedure projected to T_g gives a linear map on the cotangent space to T_g at any specified base point which transforms compatibly with base change.

Now the theorem of §2 derives from the Brouwer fixed point theorem, together with a study of possible fixed sets in the boundary sphere. A *reducible* mapping-class fixes a single boundary point; it is a foliation with closed leaves whose isotopy classes are left invariant by the diffeomorphisms in the class. A *pseudo-Anosov* class fixes two boundary points, a pair of mutually transverse foliations which have no closed leaves. There is then an *axis* in T_g, i. e. a geodesic line in the metric d defined above which joins the two fixed points and on which the mapping-class acts by translation. This motivates the appelation *hyperbolic* for these elements. The constant distortion factor K of the corresponding extremal mapping may be interpreted in various ways (see [11]) as an eigenvalue - log K is the *entropy* of the diffeomorphism.

Note. The preceding paragraph should not be viewed as more than a heuristic description or interpretation. The Teichmüller compactification described above does not admit any *continuous* extension of the Γ_g-action, by a recent result of S. Kerckhoff (Ph. D. thesis, Princeton 1978). Thurston's boundary is more natural in that sense, but involves considerably more machinery than we can conveniently handle here.

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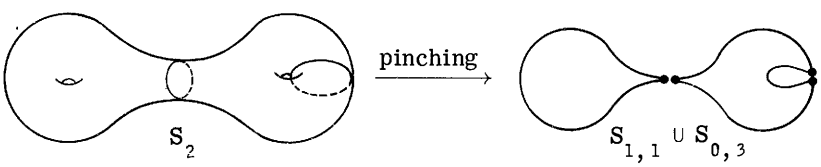
† $\mathrm{Diff}_1(S)$ is the subgroup of Diff(S) which preserves a local area element on S. See for instance [V. Arnold, Ann. Inst. Fourier, 16 (1966), 319-61].

§5. Cuspidal boundary components of T_g: a simplicial action of Γ_g

We shall examine in this section the finer boundary structure for T_g arising from a combinatorial study of the ways in which complex structures on S may degenerate. In §3, a parametrization of T_g was described with reference to a fixed set of $3g - 3$ loops in S; in this frame, allowing the $3g - 3$ lengths (or a subset) to approach 0 describes a topologically-fixed degeneration process. It is a deep result of Teichmüller space theory (also derivable from the Deligne-Mumford moduli of stable curves in algebraic geometry) that all degenerations of complex structure, properly formulated, can be described thus, modulo the operation of Γ_g, for some set of loops. More precisely, there is a collection ∂T_g of cuspidal boundary components for T_g, one for each partition (distinct set of disjoint loops) in S. Approaching such a boundary component is effected by pinching the loops to points, to obtain a set of irreducible subsurfaces with punctures, glued together by identifying pairs of punctures.



The boundary components carry natural structures of (complex) manifold compatible with intersections, such that $T_g \cup \partial T_g$ projects, modulo the operation by Γ_g, to a compact space $\tilde{\mathcal{X}}_g$ containing the moduli space $\mathcal{X}_g$ of §3. It is known in fact that $\tilde{\mathcal{X}}_g$ is a projective variety.

Notice that the operation of Γ_g on ∂T_g is the permutation action on the space of simple loops in S, combined with actions of stability subgroups on individual components. In order to describe this coherently, we follow a procedure similar to that of Borel-Serre [6]. A simplicial complex $\mathcal{T}_g$ is constructed from the set $\mathcal{L}(S)$ of all partitions of S, by giving $\mathcal{L}$ the partial ordering of inclusion of sets:- an n-simplex of $\mathcal{T}_g$ is a partition with $n + 1$ loops whose faces are the $(n - 1)$ simplices corresponding to subpartitions with n loops; we also denote by $\mathcal{T}_g$ the geometric realisation of this abstract complex. Elementary arguments

show (see [12] for terminology) that Γ_g acts simplicially on $\mathcal{T}_g$ and the following result holds.

Proposition. *$\mathcal{T}_g$ is a connected thick chamber complex if $g \geq 2$, and the quotient $\mathcal{T}_g/\Gamma_g$ is a finite complex $\mathfrak{M}$.*

The above discussion of cuspidal boundary components leads to the conclusion that ∂T_g and $\mathcal{T}_g$ are topologically equivalent in the precise sense given below.

Theorem. *The spaces ∂T_g and $\mathcal{T}_g$ have the same homotopy type. The actions of Γ_g on the two spaces are compatible with the homotopy equivalences.*

A detailed study of $\mathcal{T}_g$ and the Γ_g-action is in progress and will be published elsewhere. Two points of particular interest are mentioned below.

Notes. 1. Let Γ_Λ denote the stabiliser of a partition $\Lambda = \{l_1, \ldots, l_{3g-3}\} \subseteq \mathcal{L}$. Then there is an exact sequence

$$1 \to \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{3g-3} \to \Gamma_\Lambda \to H \to 1$$

where H is the group of automorphisms of the dual graph of the decomposed surface determined by Λ, and the free Abelian group is generated by Dehn twists about the loops in Λ. A similar structure exists for stability subgroups of non-maximal partitions, involving mapping-class groups of irreducible subsurfaces of $S \setminus \{\Lambda\}$. It can be shown that only finitely many conjugacy classes of primary elements in Γ_g fix points of $T_g \cup \partial T_g$.

2. There is a classification of elements in Γ_g, deriving from the action on $\mathcal{T}_g$, which has a similar flavour to the theorem of §2. Let $\alpha \in \Gamma_g$; the action of α on the 1-skeleton of the first barycentric subdivision of $\mathcal{T}_g$ can be lifted to an automorphism $\tilde{\alpha}$ of the universal covering tree. According to a theorem of Tits [13], either (i) $\tilde{\alpha}$ fixes a vertex or edge of the tree, or (ii) there is an infinite chain preserved by

$\tilde{\alpha}$, on which it acts by translation. The various liftings of α all fall into the same category; case (i) corresponds to reducible elements, case (ii) to hyperbolic. Elements of finite order should be excluded from the dichotomy. An interesting problem is to find an interpretation in the $\mathcal{T}_g$ framework of the eigenvalues of hyperbolic elements.

§6. Further algebraic properties of Γ_g; some open questions

The algebraic structure of Γ_g is strongly affected by the geometry of the surface S_g, and especially by the fact that $\pi_1(S_g)$ is a Fuchsian group (if $g \geq 2$) with non-trivial deformations. We shall not enlarge on that here, beyond citing some important properties derivable from this fact. The first is the result of E. Grossman that Γ_g is residually finite. Here are two easy consequences of particular value.

Proposition. (a) Γ_g is virtually torsion-free.

(b) The action of Γ_g on $T_g \cup \partial T_g$ is virtually free.

Both results follow from the finiteness of the number of conjugacy classes of maximal cyclic subgroups having non-trivial fixed sets in either T_g or ∂T_g, mentioned in §3 and in §5 note 1. We note that (a) was first proved by Serre and Grothendieck: it results by consideration of 'congruence subgroups' of Γ_g which act trivially on the homology of S when reduced modulo n ($n \geq 3$) - a result of Minkowski shows that the reduction homomorphism: $SL_N(\mathbb{Z}) \to SL_N(\mathbb{Z}/n\mathbb{Z})$ has torsion-free kernel if $n \geq 3$.

The importance of the proposition lies in the implication that the compact moduli space $\overline{\mathcal{X}}$ has a finite covering space that is a real manifold. In turn this result implies that Γ_g is finitely presented (c. f. [14, chapter 8]), a theorem due to McCool.

Here are some open questions. Many others are to be found in Birman's monograph [5].

1. Is Γ_g arithmetic? It has many arithmetic properties but no (obvious) representations.

2. What is the virtual cohomological dimension of Γ_g? Clearly we have

$$3g - 3 \leq \text{v. c. d } \Gamma_g \leq 6g - 7.$$

One suspects the answer is $3g - 3$.

3. Is there a simple geometrically-based presentation for Γ_g? Even in genus 2 (the only known one) the presentation is not terribly geometric.

4. Do the methods of this paper admit any generalisation, say to the study of Out(G) for G a finitely presented matrix group? Here it should be noted that if G is an arithmetic subgroup of an algebraic semisimple group (rank ≥ 2) then Out G is finite (A. Borel).

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For a fuller discussion of mapping class groups and their properties, see [5, 14]. The standard source for Teichmüller's theorem is Bers [2]. More details on the structure of T_g and its generalisations may be found in [3, 14]. The only source known to me of Thurston's theorem is [11], and a set of hand-written notes from an Orsay seminar, kindly sent to me by V. Poenaru. References for results quoted without source may be found in one of [3, 5, 14].

It should be noted that a recent exposé at Séminaire Bourbaki (November 1978) by V. Poenaru gives a detailed account of Thurston's theorem.

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15 · Cohomology theory of aspherical groups and of small cancellation groups

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An aspherical group Q is a discrete group which admits an aspherical presentation $(X|R)$ [9] (p. 156). Important classes of groups are aspherical (e.g. small cancellation groups, in particular almost all Fuchsian groups, one-relator groups, knot groups).

If $Q = (X|R)$, denote by F the free group on the set X of generators, and by N the normal closure in F of the elements of R. The commutator factor group $N^{ab} = N/[N, N]$ is known to be a Q-module in a natural way. If $Q = (X|R)$ is aspherical, the so-called identity problem [6], [9], [10], [11] which is, roughly speaking, that of determining the Q-module structure of N^{ab}, has a simple solution. In fact, the following holds [9] (p. 158):

Identity Theorem. *Let $Q = (X|R)$ be aspherical, and assume that no element of R is conjugate to another or to its inverse. Then the Q-module N^{ab} decomposes as a direct sum of cyclic submodules N_r generated by the elements $r[N, N]$, $r \in R$, each defined by a single relation $\tilde{z}_r \cdot r[N, N] = r[N, N]$ with $\tilde{z}_r \in Q$ the image of the root $z_r \in F$ of $r = z_r^{q_r}$.*

Actually, a converse also holds. In [4] we prove the

Theorem. *Let $Q = (X|R)$. If the Q-module N^{ab} has the structure given in the Identity Theorem, then $(X|R)$ is aspherical and no element of R is conjugate to another or to its inverse.*

The structure of N^{ab} described in the Identity Theorem immediately gives rise to a nice small free resolution of the integers over the group ring $\mathbb{Z}Q$ [3], [6]. This leads to simple formulas for the cohomology of an aspherical group Q with coefficients in any Q-module A. In fact, for

$k \geq 3$ we get $H^k(Q, A) \cong \prod_R H^k(C_r, A)$, with C_r the cyclic subgroup of Q generated by $\tilde{z}_r$. Moreover, $\tilde{z}_r$ has exact order q_r (the exponent of $r \in R$) [4]. This has drastic consequences for the class of finite subgroups of an aspherical group. For the following (unpublished) result of Serre applies:

Theorem (Serre). Let G be a group and $\{G_i\}_{i \in I}$ a family of subgroups such that for $q \geq q_0$ the canonical map $H^q(G, M) \to \Pi H^q(G_i, M)$ is an isomorphism for every G-module M. If K is a finite subgroup of G there is $i \in I$, $g \in G$ such that $K \subset gG_ig^{-1}$ and $K \cap hG_jh^{-1} = 1$ if $j \neq i$ or if $j = i$ and $h \notin gG_i$.

A proof is reproduced in [4].

Serre's Theorem clearly yields a straightforward classification of torsion elements in an aspherical group. In particular, any small cancellation group is aspherical [4]. Hence we may classify elements of finite order in such groups. This had been an open question under the so-called non-metric conditions [8], [9].

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16 · Finite groups of deficiency zero

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The aim of this survey is to give a catalogue of known finite groups with deficiency zero. The deficiency of a finite presentation $\langle X | R \rangle$ is $|X| - |R|$, and the deficiency def G of a group G is the maximum of this number taken over all finite presentations of G. It is easy to show [21, Theorem 2.6] that finite groups have non-positive deficiency, so we are dealing with an extremal case. It was known to Schur [34] that the multiplicator M(G) of a finite group G can be generated by -def G elements, and G is called efficient [15] if M(G) needs this many generators. While not all finite groups are efficient [36], the problem remains open in the nilpotent case. For more information on this and other related problems on minimal presentations, we refer the reader to the survey [42].

The ordering of our sections is basically chronological, according to the date of the first significant appearance of groups of the corresponding type, and we approach each section modulo its predecessors, cross-referencing where appropriate. Our notational conventions are fairly standard so that, for example, (a, b) denotes the highest common factor of the integers a, b, and $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^{y}$ for x, y members of a group G. The number of invariant factors of a finite abelian group is often referred to as its rank.

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§1. Centro-polyhedral groups

The earliest recorded examples of finite, non-cyclic groups of

deficiency zero appear in the work of G. A. Miller [30], who discovered them in the course of his experiments on the von Dyck groups:

$$(a,\ b,\ c) = \langle x,\ y,\ z \,|\, x^a = y^b = z^c = xyz = 1 \rangle. \tag{1}$$

It is well known that (a, b, c) is finite if and only if

$$|1/a| + |1/b| + |1/c| > 1, \tag{2}$$

and the solutions of this inequality yield the polyhedral groups:

$$(2, 2, n) = D_n,\quad (2, 3, 3) = A_4,\quad (2, 3, 4) = S_4,\quad (2, 3, 5) = A_5,$$

the full symmetry groups of the n-dihedron, tetrahedron, octahedron (cube) and icosahedron (dodecahedron) respectively. A group of deficiency zero is obtained from (1) if either the fourth or the second "=" sign in the relations is replaced by a ", ", yielding the classes of groups $\langle a, b, c \rangle$ (the binary polyhedral groups) and $\langle a, b \,|\, c \rangle$ respectively. While for the (a, b, c) the order and the signs of a, b, c are immaterial, this is no longer the case for the modified groups, and there results a bewildering array of interesting* groups. These are dealt with exhaustively in [14], and all turn out to be cyclic central extensions of and/or by the corresponding parent groups. For example, the group

$$\langle -3,\ 5 \,|\, 2 \rangle = \langle x,\ y \,|\, x^3 y^5 = (xy)^2 = 1 \rangle$$

has the presentation

$$\langle x, y, z, t \,|\, x^2 = y^3 = z^5 = xyz = t^4,\ [x, t] = [y, t] = [z, t] = 1 \rangle,$$

and is thus a central extension of Z_4 by the binary icosahedral group $\langle 2, 3, 5 \rangle \cong SL(2, 5)$. In fact, $SL(2, 5)$ furnishes our only known example of an interesting perfect group (but see §8).

That interesting groups are pretty diverse is already apparent; virtually the only common property of the centro-polyhedral groups is that

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* A group is interesting if it is a member of the class named in the title of this article.

of being 2-generated. For example, the binary dihedral groups $\langle 2^n, 2, 2\rangle$, though nilpotent and metacyclic, have unbounded order. Being of maximal class (viz. $n + 1$), they also provide examples of interesting groups of unbounded nilpotency class. Then again, centro-polyhedral groups are not all soluble, though A_5 is the only non-abelian composition factor to occur, and all the soluble ones have derived length at most three.

§2. Metacyclic groups

The groups $\langle 2^n, 2, 2\rangle$, sometimes called generalized quaternion groups, were shown to have trivial multiplicator by Schur [34], along with the quasi-dihedral groups and a further class of metacyclic p-groups. All these groups, and also groups of square-free order, were shown to have deficiency zero by B. H. Neumann [31]. This work has now been extended (using Wall's resolution [37] and the Lyndon-Hochschild-Serre spectral sequence [18] respectively) by J. W. Wamsley [39] and F. R. Beyl [2], who solved independently the efficiency problem for metacyclic groups; we paraphrase the latter version below.

A typical finite metacyclic group has the form

$$G = \langle x, y \,|\, x^m = 1, \ y^{-1}xy = x^r, \ y^n = x^s\rangle, \tag{3}$$

where $r^n \equiv 1 \pmod m$ (since y^n commutes with x), and $s = \lambda m/(m, r-1$ for some integer $\lambda \neq 0$ (since y commutes with x^s). The first condition ensures that if

$$hm = (m, r-1)(m, 1 + r + \ldots + r^{n-1}),$$

then h is an integer, and Beyl's first point is that the isomorphism class of G is not affected if we replace λ by (h, λ). His main result asserts that this number is the order of $M(G)$, and it only remains to find two relations that define G in the case when $s = m/(m, r-1)$. This is done very prettily as follows.

Let $(m, r-1) = u(r-1) + vm$, so that u is prime to $s = m/(m, r-1)$. If l is the greatest factor of m prime to u, and $t = u + ls$, then

$$(m, r-1) \equiv t(r-1) \pmod m, \text{ and } (m, t) = 1. \tag{4}$$

Since

$$[y,\ x^{-t}] = (y^{-1}x^{t}y)x^{-t} = x^{(r-1)t} = x^{(m,\,r-1)},$$

the following relations hold in G:

$$y^{n} = x^{s},\quad [y,\ x^{-t}] = x^{(m,\,r-1)}. \tag{5}$$

To see that these define G, note first that they imply that x^{s} commutes with y, and $[y,\ x^{-t}]$ commutes with x. Hence,

$$x^{ts} = y^{-1}x^{ts}y = (y^{-1}x^{t}y)^{s} = ([y,\ x^{-t}]x^{t})^{s} = [y,\ x^{-t}]^{s}x^{ts},$$

that is, $[y,\ x^{-t}]^{s} = 1$. Together with (5), this yields $x^{m} = 1$. Now by (4), there is an integer k with $kt \equiv 1 \pmod{m}$, and we have

$$[y,\ x^{-1}] = [y,\ x^{-kt}] = ([y,\ x^{-t}]x^{t})^{k}x^{-kt} = [y,\ x^{-t}]^{k} = x^{k(m,\,r-1)} = x^{kt(r-1)} = x^{r-1},$$

showing that the original relations all follow from (5).

§3. Three-generator groups

The first examples of interesting groups needing three generators were provided by J. Mennicke [29] in 1959, who showed that the groups

$$M(a,\ b,\ c) = \langle x,\ y,\ z \,|\, y^{-1}xy = x^{a},\ z^{-1}yz = y^{b},\ x^{-1}zx = z^{c}\rangle$$

are finite in the case $a = b = c \geq 3$. These groups have also been investigated by I. D. Macdonald and by J. W. Wamsley (see [38] for a detailed treatment). They are finite whenever $|a|,\ |b|,\ |c| \geq 3$, and 3-generated provided $a - 1$, $b - 1$, $c - 1$ have a common prime factor. We demonstrat below that they are finite and soluble, assuming for convenience that a, b, c are all at least 3.

First note that the defining relations imply that

$$y^{-u}x^{v}y^{u} = x^{va^{u}}, \tag{6}$$

for any integers u, v with $u \geq 0$, together with two cyclic permutants. The Witt identity

$$[x,\ y,\ z^x][z,\ x,\ y^z][y,\ z,\ x^y] = 1$$

thus yields the relation

$$x^{a^b-a}y^{b^c-b}z^{a^{-c}} = 1,$$

so that

$$\begin{aligned} x^{a^b-a}y^{b^c-b} &= z^{-1}(x^{a^b-a}y^{b^c-b})z \\ &= x^{a^b-a}.(x^{-(a^b-a)}z x^{a^b-a})^{-1}z.(y^{b^c-b})z \\ &= x^{a^b-a}.(\text{power of } z).y^{b(b^c-b)}, \end{aligned}$$

using (6). It follows that $y^{(b-1)(b^c-b)}$ is a power of z, and thus, so is $y^{(b-1)(b^{c-1}-1)}$ (conjugating by z^{-1}), whence $y^{(b-1)^2(b^{c-1}-1)} = 1$ (conjugating by z). Similarly, x and z have finite order, and since the defining relations may be used to collect powers of x, y, z in an arbitrary word, we have that

$$|M(a,\ b,\ c)| \leq (a-1)^2(a^{b-1}-1)(b-1)^2(b^{c-1}-1)(c-1)^2(c^{a-1}-1).$$

Turning to the question of solubility, note first that G' is the normal closure of x^{a-1}, y^{b-1}, z^{c-1}. Since for example, y^{b-1} commutes with y, is conjugated by z to a power of itself, and

$$x^{-1}y^{b-1}x = y^{b-1}x^{1-a^{b-1}} \in \langle y^{b-1},\ x^{a-1}\rangle,$$

it follows that G' is actually generated by these three elements. Similarly G'' is the normal closure in G' of $[y^{b-1},\ z^{c-1}] = y^{(b-1)(b^{c-1}-1)}$ and two similar elements, and since $y^{(b-1)(b^{c-1}-1)}$ is a power of z (see above), G'' is actually generated by these elements and is abelian. This shows that the derived series of G has length at most three and that its factors are all 3-generated; in particular, the Mennicke groups are all soluble.

Similar arguments are used to show that two further classes of groups (Wamsley [38], [40])

$$W_1(a, b, c) = \langle x, y, z \mid x^z = x^a,\ y^{z^{-1}} = y^b,\ z^c = [x, y] \rangle,$$

$$W_2(a, b, c) = \langle x, y, z \mid x^z = x^a,\ y^z = y^b,\ z^c = [x, y] \rangle$$

are also finite and soluble, provided $(a - 1)(b - 1)c \neq 0$.

Finally, consider the groups

$$J(a, b, c) = \langle x, y, z \mid x^y = y^b x^{-1} y^{b+4},\ y^z = z^c y^{-1} z^{c+4},\ z^x = x^a z^{-1} x^{a+4} \rangle,$$

where a, b, c are even integers distinct from -2. It turns out that the subgroup $\langle x^2, y^2, z^2 \rangle$ is finite, abelian and normal, with quotient group elementary abelian of order 8. The J(a, b, c) are investigated in [22] and [44], and complete the list of known interesting groups that need three or more generators (but see §8).

§4. Nilpotent groups

The nilpotent case is a particularly propitious one from our point of view; the multiplicator of a nilpotent group is trivial if and only if the same is true of all its Sylow subgroups, and the theory of p-groups is relatively well-developed. For example, the Golod-Shafarevich theorem (see [16] and [21]) asserts that if a finite p-group G needs d generators, then M(G) needs at least $[d^2/4] + 1$, and we deduce that all interesting nilpotent groups are at most 3-generated. Theorem 9 of [24] contains the stronger assertion that if G is any interesting group, then G/G' is at most 3-generated. Other useful properties of interesting p-groups are to be found in [20], [25], [43], for example.

Several of the groups already mentioned are nilpotent. The nilpotent centro-polyhedral groups are precisely the derivatives of $(2^n, 2, 2)$, while the metacyclic group (3) is nilpotent if and only if r - 1 involves all the prime factors of m. Criteria for the nilpotency of M(a, b, c) are harder to find, but we know that M(3, 3, 3) is a 2-group and M(-2, -2, -2) is a 3-group, and that exactly four of Wamsley's groups W_1, W_2 have prime-power order. Thus we have examples of interesting p-groups of arbitrary class for all primes p, though in all cases the central factors are at most 3-generated.

The groups

$$\mathrm{Mac}(a,\ b) = \langle x,\ y \mid x^{[x,y]} = x^{a},\ y^{[y,x]} = y^{b} \rangle$$

$a,\ b \neq 1$, were introduced in 1962 by I. D. Macdonald [27], who showed them to be nilpotent of class at most 8. The largest known nilpotency class for these groups is 7, which occurs in Mac(34, 7) [28]. Mac(a, b) has order dividing $27(a-1)(b-1)(a-1,\ b-1)^{8}$, and is the group $W_1(a,\ b,\ 1)$ of §3. By the above remarks, the Sylow p-subgroups of Macdonald groups also have trivial multiplicator, and are interesting (that is, efficient) in all cases, except possibly when $p = 2$ [45].

All groups of order 2^n and 3^n are known to be efficient for $n \leq 6$ ([26], [32], [43]). The even case yields just one non-metacyclic interesting group, namely group number 240 in [17], with presentation $W_4(-3,\ 4,\ 2)$ given below, while in the odd case, we obtain the four groups $\Delta_6(221)a$, $\Delta_6(321)a_1$, $\Delta_6(321)a_2$, $\Delta_6(221)c$ in the nomenclature of [19]. The first three of these are isomorphic to Mac(a, b), with $(a,\ b) = (-2,\ 4)$, $(-8,\ 4)$, $(-8,\ -2)$, respectively. Apart from the 2-generator 2-relation group M given in the next paragraph, and some groups in §7, this completes the list of known interesting nilpotent groups.

§5. Cyclically presented groups

A group is called cyclically presented if it has a presentation on generators $x_1, \ldots, x_n$ with n relations obtained from a single word w in the x_i by permuting the subscripts modulo n via the powers of the permutation $\theta = (1\ 2 \ldots n)$. The resulting group, denoted by $G_n(w)$, has non-negative deficiency, and thus is interesting whenever it is finite. Examples are the groups $\langle 2,\ 2,\ 2 \rangle = G_2(x_1x_2x_1x_2^{-1})$ of §1, M(a, a, a), J(a, a, a) of §3, Mac(a, a) of §4 and $\mathrm{GL}(2,\ 3) = G_3(x_1x_2x_3x_2^{-1})$. The split extension $E_n(w)$ of $G_n(w)$ by Z_n, with action induced by θ, is a 2-generator group of non-negative deficiency. The result of applying this to $Q_8 = G_3(x_1x_2x_3^{-1})$ gives $\langle 2,\ 3,\ 3 \rangle$, while applying it to M(-2, -2, -2) gives the nilpotent group M mentioned at the end of §4.

The special case $w = x_1 \ldots x_r x_{r+1}^{-1}$ (subscripts modulo n) yields the Fibonacci group $\underset{\sim}{F}(r,\ n)$ of [24], though small cancellation arguments

[13] show that these are often infinite. All of those known to be finite are in fact metacyclic, with the notable exception of the group $\underset{\sim}{F}(3, 6)$ of order 1512 [7] which is soluble of class 4, the factors of the derived series having ranks 2, 1, 2 and 1 (in descending order). A group with similar presentation and the same order as $\underset{\sim}{F}(3, 6)$ is $G_6(x_1x_3x_5x_4^{-1})$ [8] which is soluble of class 3, the factors of the derived series having ranks 1, 2 and 1. Another soluble group of class 4 is $G_3(x_1x_3x_1x_2x_3^{-1})$ which has order $2^3.3^2.7^2$. As with $\underset{\sim}{F}(3, 6)$ extending this last-mentioned group by the cyclic permutation automorphism does not increase the derived length.

The finite $G_n(w)$ have been determined for all w of length 3 and $n \leq 6$ [23], and all but one appear in the preceding sections. The exception is the group $S(2)$ of order 56, where $S(n) = G_2(x^nyxy^{n+1})$. These groups are investigated in [35], where they are shown to be finite and metabelian. Since $S(n) \cong S(-n - 3)$, we assume $n \geq -1$, and consider the group $H(n) = E_{2n+3}(x_1^{-1}x_2x_{n+4})$. It turns out that $H(n)' = G_{2n+3}(x_1^{-1}x_2x_{n+4})$. $H(n)'/H(n)''$ is finite and at most 3-generated, and $H(n)/H(n)'' \cong S(n)$, which is thus an interesting metabelian group.

§6. Soluble 2-generator groups

Two classes of interesting soluble groups were presented by Wamsley ([41] and [44]) who showed that the groups

$$W_3(a, b, c) = \langle x, y \,|\, x^{y^{-1}} = x^a y^b,\ x^{-1}y^bx = y^{cb} \rangle$$

are finite if $a, c > 1$, and that the 2-generator groups

$$W_4(a, b, c) = \langle x, y, z \,|\, x^z = x^a,\ y^2 = x^b z^c,\ z = [x, y] \rangle$$

are finite if $|a| \neq 1$ and $c \geq 0$.

The groups $W_3(a, b, c)$ are metabelian, their derived groups being 2-generated. We give a proof of these properties. Clearly the derived group of $W_3(a, b, c)$ is the normal closure of $x^{a-1}y^b$ and $y^{b(c-1)}$, so it suffices to prove that

$$[x^{a-1}y^b,\ y^{b(c-1)}] = 1 \qquad (7)$$

and that $\langle x^{a-1}y^b,\ y^{b(c-1)}\rangle$ is normal.

It is easy to see that $y^{b(c^{a-1}-1)} = 1$ so

$$x^{-(a-1)}y^b x^{a-1} = y^{c^{a-1}b} = y^b$$

and (7) follows.

To complete the proof notice that

$$(x^{a-1}y^b)x = x^{a-1}y^{bc} = (x^{a-1}y^b)(y^{b(c-1)})$$

and

$$\begin{aligned}(x^{a-1}y^b)y^{-1} &= (x^a y^b)^{a-1}y^b \\ &= x^{(a-1)^2}(xy^b)^{a-1}y^b \\ &= x^{a(a-1)}y^{b(1+c+\ldots+c^{a-1})} \\ &= (x^{a-1}y^b)^a(y^{b(c-1)})^t\end{aligned}$$

where $t = (1 + c + \ldots + c^{a-1} - a)/(c - 1)$.

A similar argument shows that $W_4(a, b, c)$ is soluble of derived length at most 3, $W_4(a, b, c)''$ is cyclic with generator $x^{(a-1)^2}$ while $W_4(a, b, c)'/W_4(a, b, c)''$ is generated by x^{a-1} and z. The group $W_4(-3, 4, 2)$, commented on in §4, has derived group abelian of rank 2. However, we have been unable to determine whether examples exist with $W_4(a, b, c)''$ non-trivial.

Further interesting classes of groups are given by Campbell and Robertson in [11]. These are the classes

$$\overline{T}(a) = \langle x,\ y \mid xy^2x^{-1}yx^2y^{-1} = 1,\ \ xy^{a+1} = y^2x^2\rangle\ ,$$

$$T(a) = \langle x,\ y \mid xy^2xyx^2y = 1,\ \ xy^{a+1} = y^2x^2\rangle\ ,$$

and

$$X(a) = \langle x,\ y \mid xy^2xyx^2y = 1,\ \ (xyxy^{-1})^a = yx^{-1}yxy^{-1}x\rangle\ .$$

If $(a, 6) = 1$, $\overline{T}(a)$ and $T(a)$ are metacyclic. However, when $(a, 6) \neq 1$ $\overline{T}(a)$ and $T(a)$ have derived length 3 and it turns out that $\overline{T}(a)/\overline{T}(a)'$ and $\overline{T}(a)''$ are cyclic while $\overline{T}(a)'/\overline{T}(a)''$ has rank 2. Provided that $a \neq 1$ $X(a)$ has derived length 3 and the derived factors are similar to those of $\overline{T}(a)$. These three classes are related as follows: $T(a)/Z_2 \cong \overline{T}(a)$ and $T(2a)$ is a subgroup of index $2n + 3$ in $X(a)$. The cyclically presented group $G_6(x_1x_3x_5x_4^{-1})$, discussed in §5, is isomorphic to $X(2)/Z_2$.

§7. Further 2-generator classes

The presentation

$$\langle x, y \,|\, xy^2 = y^3x,\ yx^2 = x^3y \rangle$$

for the trivial group given by Fox (see Coxeter and Moser, [14]), has given rise to several generalisations [1], [9] and [33]. The interesting groups discussed in these three papers are metacyclic. Less obviously the Fox presentation inspired the definition of the first class we study in this section:

$$G(a, b, c) = \langle x, y \,|\, xy^b = y^{a-1}x^cyx,\ yx^b = x^{a-1}y^cxy \rangle$$

which is studied in [4] by C. M. Campbell (see also [3] and [6]). Several subclasses of $G(a, b, c)$ give rise to finite groups. In particular we note that the groups $G(3 - 4\lambda, 5, 4\lambda - 2)$ are finite nilpotent groups of order $2^{11}\lambda$. These groups have nilpotency class 6 and derived length 3. Another subclass, $G(1 - a, b + 1, a)$, where $a \equiv 1 \pmod{b}$, has a central Z_{2a+b} with factor the centro-polyhedral group $\langle -2, 3 \,|\, b \rangle$ discussed in §1.

Of the interesting groups known to be contained in $G(a, b, c)$, the most remarkable is $G(2, 3, -2)$ discussed by Campbell in [4]. This group is an extension of $PSL(2, 8)$ by Z_3. It is the only interesting group known to have a non-abelian composition factor other than A_5 (but see §8).

Among the $G(a, b, c)$ which contain A_5 as a composition factor are $G(-1, -1, -1)$, $G(3, 3, 1)$ and $G(-2, -2, 1)$ which are $SL(2, 5)$. So are $G(0, 3, -2)$ which is $SL(2, 5) \times Z_5$, and $G(-1, -1, 4)$ which is

an extension of a group of order 2^7 by A_5. This last-mentioned group has $G'' = SL(2, 5)$. These examples and also some soluble $G(a, b, c)$ are discussed in [4] and [6]. This class contains examples of finite soluble groups of deficiency zero having Sylow subgroups with non-trivial multiplicator; in particular the metabelian group $G(2, -3, 2)$ has its Sylow 2-subgroup as its derived group and this subgroup, of order 2^6, has non-trivial multiplicator.

The second of the classes we discuss in this section is

$$F_\lambda^{a,b,c} = \langle x, y \,|\, x^2 = y^{\lambda(a+b+c)} = xy^a xy^b xy^c \rangle,$$

which generalises a presentation suggested by H. S. M. Coxeter. The special cases of $F_0^{a,b,c}$ and $F_2^{a,b,c}$ were studied in [5] and [10] respectively (see also [12]). Put $n = a + b + c$ and $d = (a-b, b-c, \lambda n)$. $F_\lambda^{a,b,c}$ is infinite if $n = 0$ and since $F_\lambda^{a,b,c} \cong F_\lambda^{-a,-b,-c}$ we can assume that $n > 0$. We also assume that $(a, b, c) = 1$. The derived series of $F_\lambda^{a,b,c}$ has factors $Z_{(\lambda+2)n}$ and groups which we shall denote by $K_\lambda^{a,b,c}$ and G_d. The abelian group $K_\lambda^{a,b,c}$ is isomorphic to $K_0^{a,b,c}$ if λ is even and to $K_{-1}^{a,b,c}$ if λ is odd, $K_{-1}^{a,b,c}$ being the maximal elementary abelian 2-factor of $K_0^{a,b,c}$.

A property worthy of note here is that the rank of the derived factor $K_\lambda^{a,b,c}$ is unbounded. More explicitly, it is shown in [12] that given any $k \geq 1$ there exists an integer $m \leq 2^{1+k} - k - 2$ such that $K_{-1}^{1,k,m}$ has rank $k + 1$. As noted in §4, interesting groups have G/G' 3-generated. These examples show that there is no bound on the rank of G'/G''. However, there is no known example with two or more derived factors of rank > 3.

For $d \geq 6$ G_d is infinite and so in this case $F_\lambda^{a,b,c}$ is not interesting. If $d = 1$ the groups $F_\lambda^{a,b,c}$ have been proved to be finite and metabelian, so $G_1 = 1$. For the cases $d = 2, 3, 4, 5$ there is only an unproved conjecture giving the structure of G_d, but proofs exist for infinitely many examples of each value of d; $G_2 = 1$, $G_3 = Z_2$, $G_4 = Q_8$ and $G_5 = SL(2, 5)$ for these known examples. In particular $F_0^{1,1,-3} \cong GL(2, 3)$ while $F_0^{2,2,-3}$ is the unimodular group modulo 5. Other identifications of groups in the class $F_0^{a,b,c}$ with centro-polyhedral groups are discussed in [5].

In common with all the other interesting groups discussed in this survey, the maximal derived length of $F_{\lambda}^{a,b,c}$ is 4 (when $d = 4$) and whenever A_5 is a composition factor it is a homomorphic image and $SL(2, 5)$ is a normal subgroup (when $d = 5$).

§8. Conclusion

The results presented in this survey suggest that answers to the following three questions would give considerable insight into the structure of finite groups of deficiency zero.

Question 1. Are the central factors of a finite nilpotent group of deficiency zero 3-generated?

Question 2. Is there a bound on the maximum derived length of a finite soluble group of deficiency zero?

Question 3. Which non-abelian simple groups can occur as composition factors of finite groups of deficiency zero?

Added in proof:

1. It has recently been shown that for odd primes p, the $SL(2)$ have deficiency zero, as also does the simple group of order 504 (C. M. Campbell and E. F. Robertson, to appear in Bull. London Math. Soc.).

2. The Mennicke groups have been generalized to yield further 3-generator interesting groups by M. J. Post (Comm. in Alg. 6 (1978), 1289-96).

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17 · Aquivalenzklassen von Gruppenbeschreibungen, Identitäten und einfacher Homotopietyp in niederen Dimensionen

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1. EINLEITUNG

(a) niederdimensionale Polyeder und Gruppenbeschreibungen

Unter einer endlichen Beschreibung

$$(1)\quad \mathfrak{P} = \{a_1, \ldots, a_g \mid R_1(a_i), \ldots, R_h(a_i)\}$$

einer Gruppe π durch die Erzeugenden a_i und die definierenden Relationen R_j verstehen wir, dass π isomorph zu $F(a_i)/N(R_j)$ ist, wobei $F(a_i)$ eine von den a_i frei erzeugte (freie) Gruppe ist und $N(R_j) \subseteq F(a_i)$ der kleinste Normalteiler, der die R_j enthält. Die Folge der R_j darf dabei auch Wiederholungen enthalten. $N(R_j)$ bleibt ungeändert, wenn auf die R_j eine endliche Folge der Elementaroperationen

(A) $R_{j_0} \to w R_{j_0} w^{-1}$ für ein j_0 und ein Wort w in den a_i (Konjugation; die übrigen R_j bleiben erhalten),

(B) freie Transformationen unter den R_j

angewandt wird. Eine solche Folge heisst nach Rapaport [13] Q-Transformation der Beschreibung (1). Sind als Elementaroperationen zusätzlich

(C) freie Transformationen unter den Erzeugenden

von $F(a_i)$ zugelassen, d. h. in den Relationen werden die a_i durch freie Transformierte ersetzt, so sprechen wir von Q*-Transformationen. Für Q**-Transformationen lassen wir ferner zu, dass

(D) eine neue Erzeugende a und eine neue Relation $R = a$ eingeführt werden (Verlängern),

und, falls möglich, der inverse Prozess $(D)^{-1}$ vorgenommen werden darf. Zwei Beschreibungen einer Gruppe π (bzw. isomorpher Gruppen π, π'), die durch eine Q**-Transformation auseinander hervorgehen, mögen Q**-äquivalent heissen. Die Äquivalenzklasse von $\mathfrak{P} = \{a_i \mid R_j\}$ bezeichnen wir mit $\phi(\mathfrak{P}) = \phi(a_i \mid R_j)$.

Äquivalente Gruppenbeschreibungen $\mathfrak{P}$ und $\mathfrak{P}'$ können stets durch eine solche Q**-Transformation ineinander überführt werden, bei welcher die Elementaroperationen in der Reihenfolge

(2) $\mathfrak{P} \xrightarrow{\text{(D)-Schritte}} \mathfrak{P}_1 \xrightarrow{\text{ein (C)-Schritt}} \mathfrak{P}^* \xrightarrow{\text{(A), (B)-Schritte}} \mathfrak{P}_1' \xrightarrow{(D)^{-1}\text{ Schritte}} \mathfrak{P}'$

ausgeführt werden, d. h. $\mathfrak{P}$ und $\mathfrak{P}'$ werden nach geeignetem Verlängern Q*-äquivalent bzw. mit einer zusätzlichen Erzeugendentransformation sogar Q-äquivalent.

Ist $\pi = 1$ die triviale Gruppe, so ergibt sich, dass die (C)-Schritte durch solche vom Typ (A), (B), $(D)^{\pm 1}$ ersetzt werden[1] können. $\mathfrak{P}$ und $\mathfrak{P}'$ sind dann nach geeignetem Verlängern sogar Q-äquivalent:

(3) $\mathfrak{P} \xrightarrow{\text{(D)-Schritte}} \mathfrak{P}_1 \xrightarrow{\text{(A), (B)-Schritte}} \mathfrak{P}_1' \xrightarrow{(D)^{-1}\text{Schritte}} \mathfrak{P}'$.

Einer endlichen Gruppenbeschreibung (1) kann man bekanntlich ein zusammenhängendes, kompaktes Polyeder K der Dimension ≤ 2 mit $\pi_1(K) \approx \pi$ zuordnen, bei dem die Relationen 2-Zellen bestimmen, die gemäss R_j in eine Rosette aus g Schlaufen geklebt sind. Umgekehrt bestimmt jedes zusammenhängende, kompakte Polyeder K der Dimension ≤ 2 eine Äquivalenzklasse von Gruppenbeschreibungen, nämlich über die Ablesungen seiner Fundamentalgruppe bezüglich semilinearer Zellaufteilungen. Diese K zugeordnete Klasse werde mit $\phi(K)$ bezeichnet.

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(1) Eine freie Erzeugendentransformation lässt sich immer aus (A), (B)-Schritten und dem Prozess des erweiterten Verlängerns (und seinem Inversen) gewinnen, bei dem eine neue Erzeugende a und eine neue Relation $R = w^{-1}a$ für ein Wort w in den alten a_i eingeführt werden dürfen. Im Falle $\pi = 1$ ist $w(a_i)$ ein Konjugiertenprodukt der alten Relationen, das erweiterte Verlängern also in einen (D)-Schritt und (A), (B)-Schritte auflösbar, vergl. Wright [19], S. 168 f.

Perrin Wright hat in [19] die Äquivalenzklassen von Gruppenbeschreibungen mit dem einfachen Homotopietyp durch folgenden Satz in Beziehung gebracht:

(4) $\phi(K) = \phi(L)$ gilt genau dann, wenn K, L durch formale p. l. - Deformationen ineinander überführt werden können, bei denen nur Erweiterungen der Dimensionen ≤ 3 auftreten.

Die Zuordnung $K \to \phi(K)$ vermittelt also eine $1:1$ Beziehung zwischen den Klassen des durch diese Dimensionsbedingung eingeschränkten einfachen Homotopietyps bei zusammenhängenden, kompakten Polyedern der Dim ≤ 2 und den Äquivalenzklassen von Gruppenbeschreibungen. (2)

Durch (4) lassen sich geometrische Fragen in algebraische übersetzen und umgekehrt. Zum Beispiel haben Andrews und Curtis in [1] die Frage gestellt, ob für zwei Beschreibungen $\mathfrak{P} = \{a_1, \ldots, a_g; R_1, \ldots, R_g\}$ und $\mathfrak{P}' = \{a'_1, \ldots, a'_{g'}; R'_1, \ldots, R'_{g'}\}$ der trivialen Gruppe stets $\phi(\mathfrak{P}) = \phi(\mathfrak{P}')$ gilt, ob also nur die triviale Klasse ϕ_0 vorkommt, die etwa durch eine triviale Beschreibung $\mathfrak{P} = \{a_1, \ldots, a_g \mid R_1 = a_1, \ldots, R_g = a_g\}$ repräsentiert wird. Die Andrews-Curtis Vermutung, dass dies zutrifft, ist nach (4) äquivalent dazu, ob bei kompakten, zusammenziehbaren Polyedern der Dimension ≤ 2 nur die Klasse des eingeschränkten einfachen Homotopietyps vorkommt, die durch kollabierbare K (etwa einen Punkt) gegeben ist.

Die Frage von Wall [17], ob der eingeschränkte einfache Homotopietyp eine echte begriffliche Einschränkung ist (s. u. 10), lautet nach (4) umgekehrt, ob aus $K \nearrow L$ stets $\phi(K) = \phi(L)$ folgt.

Es sei noch auf zwei weitere Sachverhalte hingewiesen, die sich mit (4) begründen lassen, und welche die vorliegende Arbeit motiviert haben:

I. Einer geschlossenen, zusammenhängenden 3-dimensionalen Mannigfaltigkeit M^3 kann man folgendermassen eine Beschreibungsklasse $\phi(M^3)$ zuordnen: Man entferne aus M das Innere einer semilinearen Vollkugel

(2) Der Satz von P. Wright lässt sich auch auf CW-Komplexe K ausdehnen. Ausserdem ergibt sich, dass $\phi(K)$ eine topologische Invariante ist, obwohl bei 2-dim. CW-Komplexen die 'Hauptvermutung' bereits falsch ist. Da die Beweise technische Komplikationen enthalten und wir diese Tatsachen im folgenden nicht benötigen, begnügen wir uns mit dem Hinweis darauf, s. auch Fussnote 10.

D^3 - etwa ein offenes 3-Simplex einer Triangulierung - und kollabiere den Rest, bis ein Polyeder K der Dimension ≤ 2 übriggeblieben ist.

(5) $\phi(M^3)$ werde als $\phi(K)$ definiert.

$\phi(M^3)$ ist gut definiert, denn wegen des Zusammenhangs von M ist es gleichgültig, welche Vollkugel gewählt wurde, und zwei verschiedene Kollabierungen $M^3 - \underline{D^3} \searrow K$, $M^3 - \underline{D^3} \searrow L$ erweisen K und L als zum gleichen eingeschränkten einfachen Homotopietyp gehörig. Nach (4) gilt also $\phi(K) = \phi(L)$. Es gilt sogar, dass

(6) $\phi(M^3)$ eine topologische Invariante von M^3 ist,

weil für M^3 die Hauptvermutung gilt.

Sollte es daher möglich sein, Gegenbeispiele gegen die Andrews-Curtis Vermutung zu erhalten, was ich glaube (s. u. (9)), so entsteht die äusserst wichtige Frage, ob sich geeignete Repräsentanten einer solchen nichttrivialen Klasse auf dreidimensionalen Mannigfaltigkeiten realisieren lassen. Wenn ja, wäre die 3-dim. Poincaré-Vermutung widerlegt. Überhaupt halte ich es für eine lohnende Frage, zu untersuchen, welche Beschreibungsklassen sich auf 3-Mannigfaltigkeiten (auch solchen mit Rand) realisieren lassen; dieses Problem steht 'zwischen' dem der Realisierbarkeit eines einzelnen Polyeders K^2 (s. z. B. Neuwirth [8]) und der Frage, ob eine gegebene Gruppe π als $\pi_1(M^3)$ auftritt. Lohnend erscheint die Frage ausser wegen (6) auch deshalb, weil eine Beschreibungsklasse ϕ nicht nur die übliche Homologie von Gruppen, sondern einen einfachen Homotopietyp besitzt.

II. Gegenbeispiele gegen die Andrews-Curtis Vermutung würden ferner die Vermutung von Zeeman [20] widerlegen, dass für ein kompaktes, zusammenziehbares Polyeder K der Dimension ≤ 2 $K \times I$ kollabiert; denn mit $B = I$ und $L = \{0\}$ ist es ein Spezialfall von:

(7) Gegeben seien K, L (kompakte, zusammenhängende Polyeder der Dimension ≤ 2) und ein endlicher Baum B derart, dass $K \times B$ nach L kollabiert. Dann gilt $\phi(K) = \phi(L)$.

Dies folgt nämlich nach (4) aus $K \nearrow K \times B \searrow L$. Auch hier treten interessante Fragen auf, z. B., wann aus $\phi(K) = \phi(L)$ umgekehrt die Existenz eines Baumes B mit $K \times B \searrow L$ folgt. Wenn für die triviale Klasse ϕ_0 zu $\pi = 1$ mit $B = I$ etwa $K \times I \searrow 0$ gilt, wäre nach Zeeman [20] und (6) eine Homotopiesphäre Σ^3 der Dimension 3 genau dann S^3, wenn $\phi(\Sigma^3)$ die triviale Klasse ist.

Beschreibungsklassen ϕ, ψ von π resp. π' kann man addieren, indem man Repräsentanten $\{a_i \mid R_j\}$, $\{a'_k, R'_l\}$ mit durchschnittsfremden $F(a_i)$, $F(a'_k)$ wählt und die zu $\pi * \pi'$ gehörige Klasse von $\{a_i, a'_k \mid R_j, R'_l\}$ bildet. Wenn dabei ϕ zu der geschlossenen, zusammenhängenden 3-Mannigfaltigkeit M und ψ analog zu N gehört, gilt

(8) $\phi + \psi$ ist die Klasse der zusammenhängenden Summe $M \# N$.

Für die abelsche Halbgruppe $\mathfrak{K}$ der Beschreibungsklassen mit ϕ_0 als neutralem Element drängen sich wiederum etliche Fragen auf: α) Gibt es - analog zur zusammenhängenden Summe bei 3-Mannigfaltigkeiten - eine Prim„faktor"zerlegung? β) Gilt eine Kürzungsregel? Für die Unterhalbgruppe $\mathfrak{K}^*$ der Klassen zu $\pi = 1$, die durch zusammenziehbare Komplexe (d. h. bei (1) : $g = h$) bestimmt ist: γ) Gibt es Inverse? δ) Gibt es Elemente endlicher bzw. unendlicher Ordnung? Ferner kann man zu $\mathfrak{K}$ oder einer Teilhalbgruppe $\mathfrak{K}' \subseteq \mathfrak{K}$ eine Gruppe konstruieren, indem man Klassen aus Paaren (ϕ, ψ) bildet mit $\overline{(\phi, \psi)} = \overline{(\phi', \psi')}$, wenn $\phi + \psi' + \Lambda = \phi' + \psi + \Lambda$ für ein geeignetes Λ gilt. Die Klassen werden komponentenweise addiert. Wenn in $\mathfrak{K}'$ keine Kürzungsregel gilt, bestimmt $\phi \to (\phi, \phi_0)$ (bzw. $\phi \to \overline{(\phi + \psi, \psi)}$ für $\phi_0 \notin \mathfrak{K}'$) allerdings keine Einbettung.

Vergleiche zu II. auch die Punkte b), c) und d) des Anhangs!

Aufgrund der bisherigen Erörterung ergibt sich die Aufgabe, Kriterien für die Äquivalenz von Gruppenbeschreibungen zu entwickeln. Bei isomorphen Fundamentalgruppen und gleicher Eulerscher Charakteristik sind verschiedene Homotopietypen zusammenhängender, kompakter Polyeder der Dim. ≤ 2 möglich, also erst recht: nichtäquivalente Gruppenbeschreibungen mit gleichem Wert $h - g$, s. [7]. Daher müssen

feinere Hilfsmittel herangezogen werden.

(b) Inhaltsübersicht der folgenden Paragraphen

Im zweiten Abschnitt erhalten wir solche Hilfsmittel durch den Klassifikationssatz

Satz 1. Zwei Gruppenbeschreibungen sind genau dann äquivalent, wenn sie nach geeignetem Verlängern durch einen Morphismus (s. (16)) ineinander abgebildet werden können, der sowohl a) eine Homotopieäquivalenz $K \to K'$ zugeordneter Polyeder als auch b) einen Isomorphismus der Peifferschen Identitätengruppen induziert.

Die Identitäten einer Gruppenbeschreibung, insbesondere die Peifferschen Elemente unter ihnen, wurden von R. Peiffer [12] zur algebraischen Beschreibung 3-dimensionaler Komplexe verwendet. Während in [12] die Peifferschen Elemente jedoch später wieder "herausgekürzt" werden, um den Homotopiekettenring zu erhalten, spielt der Satz von Reidemeister [14] über sie zusammen mit einem Ergebnis von Rapaport ([13], Satz 1) die Hauptrolle im Beweis von Satz 1 und erweist somit die Reidemeister-Peiffersche Theorie der Identitäten als "Eckstein" für die vorliegende Untersuchung der Äquivalenz von Gruppenbeschreibungen. Da $H_2(\tilde{K})$ isomorph zum Quotienten der Identitätengruppe nach dem von den Peifferschen Elementen erzeugten Normalteiler ist (s. (15)), kann man die Identitätengruppe als eine Art nichtabelsche zweite Homologiegruppe der universellen Überlagerung $\tilde{K}$ ansehen.[3]

Ideen zur Auswertung der Klassifikation, die durch die Identitätentheorie gegeben ist, sind im 3. Abschnitt zusammengestellt. Darunter befinden sich notwendige und hinreichende Tests, die das Hochheben von Isomorphismen betreffen. Sie lassen sich zu notwendigen Modul und Matrizenkriterien abschwächen, von denen eines für den Spezialfall der

(3) Vielleicht lohnt es sich auch, Identitäten etwa bei den Wallschen Endlichkeitsbedingungen heranzuziehen [18], wo die Dimension 2 eine Sonderrolle spielt, oder bei der Frage (s. Cohen [3], S. 81), ob endliche CW-Komplexe der Dimension 2 vom gleichen Homotopietyp auch stets denselben einfachen Homotopietyp haben.

trivialen Gruppe bereits beschrieben sei:

(9) Es existiert ein Gegenbeispiel gegen die Andrews-Curtis Vermutung, wenn gewisse aus Foxschen Ableitungen gebildete Matrizen auch dann noch nichtinvertierbar bleiben, wenn sie um beliebige Elemente aus einem Ideal von Matrizen additiv modifiziert werden dürfen, welches von den Peifferschen Elementen herrührt (Satz 5).

Die dadurch gegebene reichhaltige algebraische Situation ist solchen der algebraischen K-Theorie verwandt und der Grund dafür, warum ich glaube, dass die Andrews-Curtis Vermutung nicht zutrifft.

Die effektive Auswertung der Tests dieses Abschnitts ist das Ziel weiterer Untersuchungen, die sich an diese Arbeit anschliessen sollen. H. Ower und R. Zirpel danke ich für ihre bisherige Mühe, Teilprobleme davon in ein Maschinenprogramm zu übersetzen. Den Leser bitte ich um Mitteilung von Ideen, die den in diesem Paragraph gegebenen "Werkstatteinblick" betreffen.

Der 4. Abschnitt überträgt die Ergebnisse auf relative einfache Homotopieäquivalenzen bzw. Gruppenbeschreibungen mit Operatoren. Dass dabei nicht nur notwendige Kriterien, sondern ebenfalls Klassifikationen entstehen (Satz 1a), gelingt aufgrund eines relativen Nielsenschen Satzes (Satz 7) über Operatorbasen und -automorphismen von freien Gruppen mit gewissen Operatorgruppen, dessen Beweis kürzlich unabhängig von dieser Arbeit auch von W. Browning [2] geführt wurde. Der schon erwähnte Satz von Rapaport [13] ist ein Spezialfall davon. Ferner hat dieser Satz über relative Nielsentransformationen zur Folge, dass die Frage von Wall [17]:

(10) "$f : K \to K'$ sei eine einfache Homotopieäquivalenz zwischen zusammenhängenden CW-Komplexen. Dabei sei L ein gemeinsamer Teilkomplex, der unter f punktweise fest bleibt. $K - L$ und $K' - L$ seien endlich mit $n = \max(\dim K - L, K' - L) \leq 2$. Ist dann f homotop rel. L zu einer formalen Deformation $g : K \to K'$, bei der höchstens $n + 1$ - dimensionale Erweiterungen nötig sind?"

für $n \leq 1$ und zusammenhängende L positiv entschieden ist. Die Voraussetzungen können sogar noch abgeschwächt werden (Satz 8). Für $n = 2$ rechne ich wegen der von $\pi_1(L)$ herrührenden Operatorstruktur im relativen Fall dagegen noch stärker mit Beispielen, bei denen 3-dimensionale Erweiterungen nicht ausreichen, als im absoluten (s. u. (45)).

Ausserdem wird im 4. Abschnitt mittels Identitäten ein Test für das Wh*-Problem (Cohen [4], Metzler [7]) angegeben, welche Elemente aus $Wh(\pi)$ sich durch 2-dimensionale Erweiterungen realisieren lassen.

Über einige weitere Ergebnisse, die mit der Thematik dieser Arbeit zusammenhängen, wird überblicksartig im Anhang referiert.

R. C. Lyndon und M. M. Cohen danke ich für wertvolle Gespräche bei der Vorbereitung dieser Arbeit und die Ermutigung, sie niederzuschreiben, obwohl z. B. der 3. Abschnitt den Wunsch nach weiteren Resultaten weckt. Obgleich die Ergebnisse nicht plagiiert sind, danke ich insbesondere für ein Gesprächsklima, bei dem Freude über die Arbeit an den sich offenbarenden Zusammenhängen und nicht Sorge um Priorität bei den Resultaten bestimmend war.

Wie mir während der Niederschrift des Manuskriptes bekannt wurde, hat R. Craggs [5] eine Arbeit über verwandte Fragen vorbereitet. Auf sie möchte ich abschliessend hinweisen.

2. IDENTITÄTEN

Einer endlichen Gruppenbeschreibung $\mathfrak{P} = \{a_1, \ldots, a_g \mid R_1(a_i), \ldots, R_h(a_i)\}$ ordnen wir die folgenden zusätzlichen Daten zu: Wir bilden die freie Gruppe $H \supseteq F(a_i)$ mit den freien Erzeugenden $a_1, \ldots, a_g, r_1, \ldots, r_h$ und erklären einen Homomorphismus $p : H \to F(a_i)$ durch $p(a_i) = a_i$, $p(r_j) = R_j$. Das p-Bild eines Elementes $x \in H$ bezeichnen wir gemäss Reidemeister [14] auch als $\dot{x}$. In H erzeugen die r_j einen Normalteiler H^*, der unter p nach $N(R_j)$ abgebildet wird. Der Kern von $p|H^*$ werde als Identitätengruppe I bezeichnet. Sie beschreibt die Abhängigkeiten zwischen den Relationen R_j : zwei "formal verschiedene" Konjugiertenprodukte in den R_j, die dasselbe Element von $N(R_j)$ darstellen, stammen von verschiedenen Elementen aus H^*, deren Quotient in I liegt - und umgekehrt. Die Identitäten der

Form $(r, s) = r.s.r^{-1}.\dot{r}s^{-1}\dot{r}^{-1}$ für r, s aus H^* heissen Peiffersche Elemente; der von ihnen in H erzeugte Normalteiler P ist die Peiffersche Gruppe. Insgesamt haben wir die Reihe $H \supseteq H^* \supseteq I \supseteq P$ von H-Normalteilern erhalten.

Mit der Geometrie des $\mathfrak{P}$ zugeordneten Komplexes K der Dimension ≤ 2 stehen diese Daten in folgender Beziehung: Es existiert das kommutative Diagramm

$$(11) \qquad \begin{array}{ccc} H^* & \xrightarrow{f} & C_2(\tilde{K}) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \partial} \\ N(R_j) & \xrightarrow{e} & C_1(\tilde{K}) \end{array} ,$$

wobei auf der rechten Seite die Kettengruppen und der Randoperator der universellen Überlagerung $\tilde{K}$ stehen. p^* stimmt mit p bis auf den Argument - und ggf. Bildbereich überein. e ist durch die Foxschen Ableitungen (mit Werten im Gruppenring $\mathbb{Z}(\pi_1(K))$) gegeben und erfüllt

$$(12) \qquad \mathrm{Kern}(e) = [N(R_j), N(R_j)] = p^{(*)}([H^*, H^*]).$$

f ordnet einem Produkt $\prod_k w_k(a_i) r_{j_k}^{\varepsilon_k} w_k(a_i)^{-1}$ mit $\varepsilon_k = \pm 1$ die Summe $\sum_k \varepsilon_k g_k(\tilde{e}^2_{j_k})$ zu, wobei die g_k die nach π_1 projezierten w_k sind und die $\tilde{e}^2_j$ einen Fundamentalbereich von 2-Zellen in $\tilde{K}$ bilden.

Nach Reidemeister [14] gilt:

$$(13) \qquad \begin{array}{ll} \text{(a)} & f \text{ ist epi mit Kern } N([H^*, H^*] \cup P) \\ \text{(b)} & I \cap N([H^*, H^*] \cup P) = P. \end{array}$$

Ein Element von I wird unter f wegen der Kommutativität des Diagramms (11) in einen 2-Zyklus abgebildet.

(14) Jeder 2-Zyklus ist aber auch in $f(I)$ enthalten,

denn für $z \in C_2(\tilde{K})$ mit $\partial(z) = 0$ gibt es nach (13a) ein $x \in H^*$ mit $f(x) = z$. Aus $e\,p^{(*)}(x) = \partial f(x) = \partial z = 0$ folgt dann wegen (12): $p(x) \in p([H^*, H^*])$. x ist also von einem geeigneten $k \in [H^*, H^*]$ um eine Identität $i \in I$ verschieden: $x = k \cdot i$. Unter f wird k wegen (13a)

nach 0 abgebildet, so dass wir

$$z = f(x) = f(k \cdot i) = f(i) \quad \text{erhalten}^{(4)}, \quad \text{q. e. d.}$$

Weil I unter f auf $H_2(\tilde{K})$ abgebildet wird und dabei nach (13a, b) der Kern P auftritt, gilt:

(15) f induziert einen Isomorphismus $I/P \to H_2(\tilde{K})$.

Sind $\mathfrak{P} = \{a_1, \ldots, a_g; R_1, \ldots, R_h\}$, $\mathfrak{P}' = \{a_1, \ldots, a_g; R_1', \ldots, R_{h'}'\}$ zwei Beschreibungen der Gruppen π resp. $-\pi'$ mit denselben Erzeugenden und den Daten p, $H \supseteq H^* \supseteq I \supseteq P$ resp. p', $H' \supseteq H'^* \supseteq I' \supseteq P'$, so verstehen wir unter einem Morphismus von $\mathfrak{P}$ nach $\mathfrak{P}'$ einen Homomorphismus $\rho : H \to H'$ mit den Eigenschaften:

(16) (a) $\rho | F(a_i)$ bestimmt einen Isomorphismus μ von $F(a_i)$ auf sich: $\rho | F(a_i) : F(a_i) \xrightarrow{\mu} F(a_i) \hookrightarrow H'$.

(b) $\rho(H^*) \subseteq H'^*$

(c) das Diagramm
$$\begin{array}{ccc} H & \xrightarrow{\rho} & H' \\ \downarrow p & & \downarrow p' \\ F & \xrightarrow{\mu} & F \end{array}$$
kommutiert.

ρ bildet dann I nach I' und P nach P' ab. Die Komposition zweier Morphismen ist wieder ein solcher. Der identische Morphismus von $\mathfrak{P}$ auf sich ist durch $\rho = id_H$, ($\Leftrightarrow \mu = id_F$) festgelegt. Ein Morphismus ist genau dann ein Isomorphismus, d. h. besitzt einen inversen Morphismus, wenn $\rho : H \to H'$ isomorph ist.[5] Für isomorphe $\mathfrak{P}$, $\mathfrak{P}'$ gilt: $h = h'$.

(17) Sind
$$\begin{array}{ccc} H & \xrightarrow{\rho_0} & H' \\ \downarrow p & & \downarrow p' \\ F & \xrightarrow{\mu} & F \end{array}$$
und
$$\begin{array}{ccc} H & \xrightarrow{\rho_1} & H' \\ \downarrow p & & \downarrow p' \\ F & \xrightarrow{\mu} & F \end{array}$$
zwei Morphismen,

die dasselbe μ induzieren, so unterscheiden sie sich um Identitäten, d. h. es gilt: $\rho_1(r_j) = \rho_0(r_j) \cdot i_j'$ mit $i_j' \in I'$; umgekehrt ergibt sich

(4) Geometrisch lässt sich (14) beweisen, indem man ausnutzt, dass in $\tilde{K}$ jeder 2-Zyklus sphärisch ist. Eine geeignete zelluläre Abbildung $S^2 \to \tilde{K}$ ergibt dann eine Relationenidentität, die den vorgegebenen Zyklus repräsentiert.

(5) Insbesondere führt die Auswahl von anderen Erzeugendensymbolen statt der r_j bei der Konstruktion von $H \xrightarrow{p} F$ natürlich zu einer isomorphen Situation.

so durch Wahl der $i'_j \in I'$ aus ρ_0 stets ein ρ_1 mit demselben μ.

Der folgende Sachverhalt bringt Morphismen mit Q*-Transformationen in Beziehung:

(18) $\mathfrak{P}$ und $\mathfrak{P}'$ sind genau dann isomorph, wenn sie Q*-äquivalent sind.

Beweis von (18): Für Elementarschritte (A), (B), (C) lassen sich aus deren Definition unmittelbar Beschreibungsisomorphismen angeben, und zwar mit $\mu = \mathrm{id}_F$ für (A), (B), $\rho(r_j) = r'_j$ für (C). Daher sind Q*-äquivalente Beschreibungen isomorph. Ist umgekehrt ein Beschreibungsisomorphismus

$$\begin{array}{ccc} H & \xrightarrow{\rho} & H' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ F & \xrightarrow{\mu} & F \end{array}$$

gegeben, so bilden die $\rho(r_1), \ldots, \rho(r_h)$ und die $r'_1, \ldots, r'_h$ zwei Operatorbasen von H'^* bezüglich der Konjugation mit Elementen aus F, d. h. die $w_k \rho(r_j) w_k^{-1}$ resp. $w_k r'_j w_k^{-1}$ bilden eine Basis im gewöhnlichen Sinn, wenn w_k die Elemente von F durchläuft. Zwei solche Operatorbasen können nach Rapaport ([13], Satz 1) oder dem relativen Nielsenschen Satz (Satz 7) durch eine Folge aus freien Transformationen und Konjugationsschritten mit Konjugatoren $w \in F$ ineinander überführt werden. Daraus ergibt sich, dass $\mathfrak{P}'$ aus $\mathfrak{P}$ durch einen (C)-Schritt und anschliessende (B)- und (A)-Schritte hervorgeht, q. e. d.

Ein Beschreibungsmorphismus ρ von $\mathfrak{P}$ nach $\mathfrak{P}'$ induziert eine zelluläre stetige Abbildung $K \to K'$ der zugeordneten polyedrischen Komplexe der Dimension ≤ 2. Dabei bestimmt ρ vermöge $F/p(H^*) \to F/p'(H'^*)$ den Fundamentalgruppenhomomorphismus und nach (15) vermöge $I/P \to I'/P'$ den Homomorphismus $H_2(\tilde{K}) \to H_2(\tilde{K}')$. Ist ρ ein Isomorphismus von $\mathfrak{P}$ nach $\mathfrak{P}'$, so werden P, I, H^*, H isomorph auf die entsprechenden $\mathfrak{P}'$-Daten abgebildet; daher entsteht eine Abbildung $K \to K'$, die Isomorphismen der Fundamentalgruppen und der zweiten Homologiegruppen der universellen Überlagerungen induziert. ρ ergibt also

(19) (a) eine Homotopieäquivalenz $K \to K'$ und

(b) einen Isomorphismus $P \to P'$.

ρ möge jetzt umgekehrt diese Eigenschaften (a) und (b) haben. Dann induziert ρ

(20) Isomorphismen
$$\begin{aligned} H/H^* &\to H'/H'^* \\ H^*/I &\to H'^*/I' \\ I/P &\to I'/P' \\ P &\to P' ; \end{aligned}$$

denn die oberste Zeile folgt, weil F isomorph auf sich abgebildet wird, was wir von jedem Morphismus verlangen; die zweite Zeile bedeutet, dass $N(R_j)$ isomorph auf $N(R_j')$ abgebildet wird, und die dritte, dass $H_2(\tilde{K}) \to H_2(\tilde{K}')$ ein Isomorphismus ist; die Isomorphismen der 2. und der 3. Zeile folgen also aus (a); (b) ergibt den der 4. Zeile. Nun erhalten wir für die H-Normalteilerreihe $H \supseteq H^* \supseteq I \supseteq P$ und die ensprechenden $\mathfrak{P}'$-Daten aus (20) sukzessive von unten nach oben, dass ρ Isomorphismen $I \to I'$, $H^* \to H'^*$ und $H \to H'$ bestimmt, d. h. ein Beschreibungsisomorphismus ist. Wir haben also gezeigt:

(21) Ein Beschreibungsmorphismus ρ von $\mathfrak{P}$ nach $\mathfrak{P}'$ ist genau dann ein Isomorphismus, wenn er (19) erfüllt.

Die Punkte (18) und (21) fassen wir zusammen zu dem

Lemma. Zwei Gruppenbeschreibungen $\mathfrak{P}, \mathfrak{P}'$ sind genau dann Q*-äquivalent, wenn es einen Beschreibungsmorphismus ρ von $\mathfrak{P}$ nach $\mathfrak{P}'$ gibt, der sowohl

(a) eine Homotopieäquivalenz $K \to K'$ zugeordneter Polyeder als auch

(b) einen Isomorphismus $P \to P'$ induziert.

Der Beweis von Satz 1 ergibt sich aus diesem Lemma nun unmittelbar, wenn man (2) berücksichtigt.

Aus (3) folgt für $\pi = 1$ sogar

Satz 2. Zwei Beschreibungen $\mathfrak{P}$, $\mathfrak{P}'$ der trivialen Gruppe sind genau dann äquivalent, wenn sie nach geeignetem Verlängern durch einen Morphismus mit $\mu = \mathrm{id}_F$ ineinander abgebildet werden können, der (19) erfüllt.

Für eine Beschreibung $\mathfrak{P} = \{a_1, \ldots, a_g | R_1(a_i), \ldots, R_g(a_i)\}$, die Q*-äquivalent zu einer trivialen Beschreibung $\mathfrak{P}' = \{a_1, \ldots, a_g | R_1' = a_1, \ldots, R_g' = a_g\}$ von $\pi = 1$ ist, erhalten wir schärfer als (3) - nämlich ohne zusätzliches Verlängern -, dass

(22) $\mathfrak{P}$ Q-äquivalent zu $\mathfrak{P}'$ ist,

denn aus der ursprünglichen Q*-Transformation ergibt sich eine Q-Transformation der R_j zu freien Transformierten der $R_j' = a_j$, wenn man die (C)-Schritte nicht ausführt. Die (C)-Schritte lassen sich dann nachträglich durch einen (B)-Schritt ersetzen. Wenn man (22) bei (18) und (21) berücksichtigt, folgt, da (19a) für zusammenziehbare K, K' immer gilt:

Satz 3. Eine Gruppenbeschreibung $\mathfrak{P} = \{a_1, \ldots, a_g | R_1, \ldots, R_g\}$ von $\pi = 1$ ist genau dann Q*-äquivalent zu der trivialen Beschreibung $\mathfrak{P}' = \{a_1, \ldots, a_g | R_1' = a_1, \ldots, R_g' = a_g\}$, wenn es einen Morphismus ρ von $\mathfrak{P}$ nach $\mathfrak{P}'$ mit $\mu = \mathrm{id}_F$ gibt, der einen Isomorphismus $P \to P'$ induziert.

Es mögen noch einige Einsichten über die Identitätengruppe bzw. die Peiffersche Gruppe einer Beschreibung $\mathfrak{P}$ folgen, wobei wir die sukzessive eingeschränkten Fälle:

(α) $\pi = 1$, (β) $K \simeq 0$, (γ) die Klasse der Q*-trivialen Beschreibungen betrachten.

(α) Da $N(R_j) = F(a_i)$ gilt, gibt es zu jedem a_i eine geeignete Relation $s_i \in H^*$ mit $\dot{s}_i = a_i$. Mit ihren p-Bildern sind die s_i frei; genauer: die von den s_i frei erzeugte Untergruppe $S \subseteq H^*$ wird unter p isomorph auf $F(a_i)$ abgebildet. Zu jedem $r \in H^*$ gibt es dann genau ein $\bar{r} \in S$ mit $\dot{r} = \dot{\bar{r}}$, d. h. $r \cdot \bar{r}^{-1}$ ist eine Identität. Es gilt:

(23) I wird durch die H-Konjugierten der $h + g^2$ Elemente
$r_j \cdot \bar{r}_j^{-1}$, $j = 1, \ldots, h$ und
$(s_i, s_k) = s_i \cdot s_k \cdot s_i^{-1} \cdot a_i s_k^{-1} a_i^{-1}$, $i, k = 1, \ldots, g$ erzeugt.

I kann also bezüglich H-Konjugation insbesondere durch endlich viele Elemente erzeugt werden.

Zum Beweis von (23) betrachten wir die Äquivalenzklassen, die in H^* durch die angegebenen Elemente erzeugt werden. $r_j \sim \bar{r}_j$ hat für $r \in H^*$ zur Folge, dass $r \sim r' = \prod_\nu w_\nu(a_i) s_{i_\nu}^{\varepsilon_\nu} w_\nu(a_i)^{-1}$, $\varepsilon_\nu = \pm 1$ mit einem geeigneten F-Konjugiertenprodukt r' der s_i gilt. Aus $a_i s_k a_i^{-1} \sim s_i \cdot s_k \cdot s_i^{-1}$ folgt dann sogar $r \sim r' \sim r'' \in S$. r, r' und r'' unterscheiden sich nur um Identitäten. Für $r \in I$ gilt daher auch $r'' \in I$, wegen des Isomorphismus $p|S : S \to F(a_i)$ also $r'' = 1$. Wir haben somit $r \sim 1$ erhalten, q. e. d.

(β) Aus (15) und (23) folgt wegen $H_2(\tilde{K}) = 0$ und $g = h$:

(24) $I = P$ wird durch die H-Konjugierten der $g + g^2$ Elemente
$r_j \cdot \bar{r}_j^{-1}$ und (s_i, s_k) erzeugt.

(γ) Wir betrachten zunächst eine triviale Beschreibung $\mathfrak{P} = \{a_1, \ldots, a_g | R_1 = a_1, \ldots, R_g = a_g\}$. Hier können wir $s_i = r_i$ wählen und erhalten $r_j \cdot \bar{r}_j^{-1} = 1$, so dass $I = P$ durch die H-Konjugate der (r_i, r_k) erzeugt wird. Wegen $(r_i, r_k) = r_i \cdot r_k \cdot r_i^{-1} \cdot a_i r_k^{-1} a_i^{-1}$
$= a_i \cdot ((a_i^{-1} r_i) \cdot r_k \cdot (r_i^{-1} a_i) \cdot r_k^{-1}) \cdot a_i^{-1} = a_i \cdot [a_i^{-1} r_i, r_k] \cdot a_i^{-1}$ wird P als Normalteiler in H von den Kommutatoren
$[a_i^{-1} r_i, r_k] = (a_i^{-1} r_i a_i, a_i^{-1} r_k a_i)$ erzeugt. Bezüglich der freien Erzeugenden $\alpha_i = a_i^{-1} r_i$ und r_k $i, k = 1, \ldots, g$ von H ist also die Quotientenbildung $H \to H/P$ durch die Projektion

(25) $F(\alpha_i) * F(r_k) \to F(\alpha_i) \oplus F(r_k)$

eines freien Produktes zweier freier Gruppen in deren direkte Summe gegeben. Mit Hilfe des Reidemeister-Schreier-Verfahrens z. B. ergibt sich daraus, dass der Kern P durch die $w \cdot v \cdot [\alpha_i, r_k] \cdot v^{-1} \cdot w^{-1}$ frei erzeugt wird, wenn w und v unabhängig voneinander alle r- resp. α-Kurzwörter durchlaufen. Diese Konjugate sind mit den $[\alpha_i, r_k]$ aus

P und ebenso ihre Bilder bei einem Beschreibungsmorphismus. Wir erhalten daher für die triviale Beschreibung $\mathfrak{P}$ und - vermöge eines Beschreibungsisomorphismus - auch für jede zu ihr $Q^{(*)}$-äquivalente:

(26) Bei Auswahl eines geeigneten Repräsentanten x_ν aus jeder Restklasse von H nach P und geeigneter g^2-vieler Peiffer-scher Elemente y_μ besitzt P die $x_\nu y_\mu x_\nu^{-1}$ als Basis.

Ist im Fall (β) eine Reduktion der Situation (24) auf (26) unmöglich, liegt daher eine $Q^{(*)}$-nichttriviale Beschreibungsklasse zu $K \simeq 0$ vor. Weitere Tests für $Q^{(*)}$- und Q**-Äquivalenz folgen im nächsten Abschnitt, ohne den Anspruch auf Vollständigkeit der Liste zu erheben. (6)

3. ÄQUIVALENZKRITERIEN

(a) Ist für zwei Gruppenbeschreibungen $\mathfrak{P} = \{a_1, \ldots, a_g | R_1, \ldots, R_h\}$ von π und $\mathfrak{P}' = \{a_1, \ldots, a_g | R_1', \ldots, R_h'\}$ von π' ein Isomorphismus $\mu : F(a_i) \to F(a_i)$ gegeben, der sich in einen Homomorphismus $\pi \to \pi'$ durchdrückt, so gibt es stets einen Beschreibungsmorphismus $\rho_0 : H \to H'$, der μ induziert: μ bildet nämlich $N(R_j)$ nach $N(R_j')$ ab, und wenn dabei R_j in ein F-Konjugiertenprodukt der R_j' übergeht, kann man $\rho_0(r_j)$ als das entsprechende F-Konjugiertenprodukt der r_j' wählen. Alle möglichen für μ zulässigen Morphismen ρ ergeben sich aus ρ_0 gemäss (17) durch Variation der $\rho_0(r_j)$ um Identitäten. Wir wollen untersuchen, wann ρ_0 zu einem für μ zulässigen Beschreibungsisomorphismus variiert werden kann. Dazu ist notwendig, dass μ sogar einen Isomorphismus von π auf π' induziert, was wir in diesem Abschnitt von nun an stets annehmen Ein zulässiges ρ induziert in der Liste (20) dann bereits durch μ festgelegte Isomorphismen von H/H^*, H^*/I, also auch von H/I auf die entsprechenden Gruppen von $\mathfrak{P}'$. Dass ρ_0 zu einem zulässigen Beschreibungsisomorphismus ρ variiert werden kann, ist gleichwertig damit, dass bei

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(6) Z.B. ergibt ein Beschreibungsisomorphismus von $\mathfrak{P}$ nach $\mathfrak{P}'$ Nielsen-äquivalente Erzeugendensysteme von H/I und H'/I' resp. H/P und H'/P' vermöge der Zuordnung $H \to H'$. Hieraus kann man versuchen, $Q^{(*)}$-Tests zu gewinnen.

(27) $\begin{array}{ccc} H/P & - - - \rightarrow & H'/P' \\ \downarrow & & \downarrow \\ H/I & \xrightarrow[\approx]{} & H'/I' \end{array}$ der durch μ gegebene Isomorphismus der unteren Zeile sich durch eine kommutative Ergänzung zu einem Isomorphismus der oberen hochheben lässt, wobei die Bilder der a_i durch μ bereits festgelegt sind,

und anschliessend unter denselben Bedingungen bei

(28) $\begin{array}{ccc} H & \overset{\rho}{- - \rightarrow} & H' \\ \downarrow & & \downarrow \\ H/P & \longrightarrow & H'/P' \end{array}$ ein Hochheben des unteren Isomorphismus zu einem der oberen Zeile möglich ist.

Alle hochgehobenen Homomorphismen $H \rightarrow H'$ mit $\rho(a_i) = \mu(a_i)$ unterscheiden sich nämlich nur in den $\rho(r_j)$ und dies um Elemente aus I'. Sie sind daher mit der Hochhebung ρ_0 zulässige Morphismen für μ.

Dass bei (27) kein Hindernis auftritt, ist gleichwertig damit, dass dabei ein Isomorphismus $I/P \rightarrow I'/P'$, also eine Homotopieäquivalenz $K \rightarrow K'$ induziert werden kann. Im allgemeinen kann (27) selbst für $h = h'$ bereits unerfüllbar sein [7] und, falls doch, (28) z. B. nur für eine bestimmte Wahl der Hochhebung bei (27). Die Entscheidung darüber, ob μ zu einem Beschreibungsisomorphismus gehört, ist also im allgemeinen kompliziert.

Sie vereinfacht sich jedoch im Falle, dass $\mathfrak{P}$ zu einem zusammenziehbaren Komplex gehört und geprüft werden soll, ob $\mathfrak{P}$ Q*-trivial ist. Dann können wir uns nach (22) auf $\mu = id_F$ beschränken, und da für $\mathfrak{P}$ bzw. die triviale Beschreibung $\mathfrak{P}' = \{a_1, \ldots, a_g \mid R'_1 = a_1, \ldots, R'_g = a_g\}$ gilt: $I' = P'$, ist bei (28) ein genau bestimmter Isomorphismus $\bar{\rho} : H/P \rightarrow H'/P'$ auf seine Hochhebbarkeit zu prüfen.

Unter Berücksichtigung von (25) für $\mathfrak{P}'$ haben wir daher gezeigt:

Satz 4. Die Beschreibung $\mathfrak{p}$ mit einem zusammenziehbaren Komplex genau dann Q*-trivial, wenn in dem Diagramm

$$\begin{array}{ccc} H & \overset{\rho}{- - \rightarrow} & F(\alpha'_i) * F(r'_j) \\ \downarrow & & \downarrow \\ H/P & \xrightarrow{\bar{\rho}} & F(\alpha'_i) \oplus F(r'_j), \quad i, j = 1, \ldots, g \end{array}$$

sich die aus den 2g Bildern der a_i, r_j bestehenden Erzeugenden von $F(\alpha_i') \oplus F(r_j')$ so zu einer Basis von $F(\alpha_i') * F(r_j')$ hochheben lassen, dass bei dem entstehenden Isomorphismus ρ jedes a_i in $a_i' = r_i' \alpha_i'^{-1}$ übergeht. (7)

Satz 4 legt es nahe, allgemein zu untersuchen,

(29) wann sich ein Erzeugendensystem der Länge $m + n$ von $F(a_i) \oplus F(b_j)$ $i = 1, \ldots, m$; $j = 1, \ldots, n$ zu einer Basis von $F(\{a_i\} \cup \{b_j\}) = F(a_i) * F(b_j)$ hochheben lässt.

Für $n = 1$ ist das immer der Fall, denn wenn $(w_1(a_i), b^{\nu_1}), \ldots, (w_{m+1}(a_i), b^{\nu_{m+1}})$ ein solches Erzeugendensystem ist, können wir die linken Koordinaten mittels Nielsenscher Reduktionen auf Normalform bringen und auf die rechten die analogen Schritte anwenden. Ohne dass sich die Eigenschaft der Hochhebbarkeit ändert, erhalten wir so

$$(a_1, b^{\nu_1'}), \ldots, (a_m, b^{\nu_m'}), \quad (1, b^{\nu_{m+1}'}) .$$

Der abelschgemachte Fall lehrt nun $\nu_{m+1}' = \pm 1$, so dass wir weitertransformieren können zu $(a_1, 1), \ldots, (a_m, 1), (1, b)$. Diese Erzeugendenmenge ist hochhebbar und somit auch die ursprüngliche.

Ein analoges Argument mit unimodularen Matrizen führt auch bei endlich vielen Summanden $F(a_i) \oplus F(b_j) \oplus F(c_k) \ldots$ zum Ziel, wenn

(30) alle Summanden bis auf höchstens einen den Rang 1 haben.

So bleibt als erster interessanter Fall bei (29) der mit $m = n = 2$; und dieser tritt bei Satz 4 z. B. für das von Osborne in [11] angegebene Beispiel einer Darstellung mit zusammenziehbarem Komplex auf, bei welcher die Q^{**} – Trivialität noch nicht geklärt ist:

(31) $\mathfrak{P} = \{a, b;\ a^{-3}b^{-1}a^2b,\ b^{-3}a^{-1}b^2a\}$

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(7) Vielleicht besteht hier eine Verbindung zu dem Faktorisierungsproblem (D) bei Jaco [6], welches äquivalent zur 3-dim. Poincaré-Vermutung ist.

Zur Behandlung von (29) bietet sich m. E. ausser Methoden der kombinatorischen Gruppentheorie auch die von Rothaus z. B. in [15] verwendete Technik an, die Erzeugenden einer freien Gruppe als Variable in einer Liegruppe L zu deuten. Sind $w_1(a_i, b_j), \ldots, w_{m+n}(a_i, b_j)$ Elemente aus $F(\{a_i\} \cup \{b_j\})$, deren Projektionen $F(a_i) \oplus F(b_j)$ erzeugen, so wird vermöge

$$(a_1, \ldots, a_m, b_1, \ldots, b_n) \to (w_1, \ldots, w_{m+n})$$

eine differenzierbare Abbildung $\rho_0 : L^{m+n} \to L^{m+n}$ gegeben. Aus der Frage (29) wird dann:

(32) Kann $\rho_0 : L^{m+n} \to L^{m+n}$ um Konjugiertenprodukte der Kommutatoren $[a_i, b_j]$ auf den Bildkoordinaten so modifiziert werden, dass ein Diffeomorphismus $\rho : L^{m+n} \to L^{m+n}$ entsteht?

Zu ihrer Untersuchung stehen nun alle (algebraisch-) topologischen Hilfsmittel der Theorie Liescher Gruppen zur Verfügung, s. [15].

(b) Wir wenden uns wieder einer beliebigen Gruppenbeschreibung $\mathfrak{P}$ zu: P ist abgeschlossen bezüglich Konjugation mit Elementen aus H, ebenso [P, P]. Daher wird die abelschgemachte Gruppe P/[P, P] zu einem H-Modul, ja sogar zu einem H/P-Modul, denn es gilt $hx \cdot x_0 \cdot x^{-1}h^{-1} = hxh^{-1} \cdot hx_0h^{-1} \cdot hx^{-1}h^{-1} \sim h \cdot x_0 \cdot h^{-1}$ für $x_0, x \in P$, $h \in H$ und die durch [P, P] in P erzeugten Äquivalenzklassen

Ist daher ein Beschreibungsmorphismus ρ_0 von $\mathfrak{P}$ nach $\mathfrak{P}'$ gegeben, der einen Isomorphismus $\bar{\rho}_0 : H/P \to H'/P'$ induziert, (also (27) erfüllt), so lässt sich die Forderung (28) folgendermassen abschwächen

(33) Die $\rho_0(r_j)$ müssen um Elemente aus P so modifizierbar sein, dass ein Beschreibungsmorphismus ρ entsteht, der den H/P-Modul P/[P, P] isomorph auf den H'/P'-Modul P'/[P', P'] abbildet.

(Bei jeder Modifikation um Elemente aus P gilt $\bar{\rho} = \bar{\rho}_0$, so dass der Isomorphismus der Operatoren festliegt.)

Analoge Kriterien lassen sich aufstellen, wenn man $P/[P, P]$ nur als H^*/P - oder $H_2(\tilde{K})$-Modul auffasst (letzteres vermöge (15)). Für I statt P lassen sich ebenfalls solche Modultests angeben.

Aus (26) folgt für eine $Q^{(*)}$-triviale Beschreibung $\mathfrak{P}$:

(34) Der H/P-Modul $P/[P, P]$ ist frei vom Rang g^2.

Bei der trivialen Beschreibung $\mathfrak{p}' = \{a_1, \ldots, a_g \mid R'_1 = a_1, \ldots, R'_g = a_g\}$ bilden gemäss den vor (26) durchgeführten Betrachtungen z. B. die Restklassen der $[\alpha'_i, r'_j]$ oder der (r'_i, r'_j) eine Basis. (33) ergibt für eine zu $\pi = 1$ gehörige Beschreibung $\mathfrak{P} = \{a_1, \ldots, a_g \mid R_1, \ldots, R_g\}$ und $\mathfrak{P}'$ speziell ein Kriterium für die Q^*-Trivialität von $\mathfrak{P}$, wobei $\bar{\rho} : H/P \to H'/P'$ durch $\mu = \mathrm{id}_F$ festgelegt ist. Dieser Test unterscheidet u. U. mehr als die abstrakten Isomorphieforderungen, dass H/P direkte Summe zweier freier Gruppen vom Rang g ist (s. (25)) und (34) gilt.

Sind zwei (beliebige) Gruppenbeschreibungen $\mathfrak{P}$, $\mathfrak{P}'$ isomorph, so entsteht eine $1:1$ Zuordnung zwischen den Morphismenmengen von $\mathfrak{P}$ nach $\mathfrak{p}$ und von $\mathfrak{P}'$ nach $\mathfrak{p}$, wenn $\mathfrak{p}$ dieselben Erzeugenden wie $\mathfrak{P}'$ hat. Hieraus lassen sich $1:1$ Zuordnungen der Homomorphismenmengen entsprechender Moduln als Q^*-Invarianztests gewinnen.

(c) Mit Hilfe Foxscher Ableitungen lassen sich die unter (b) angegebenen Modul- in Matrizenkriterien verwandeln. Wir stellen ein solches im folgenden sogleich für die Abbildung $H \to H'$ eines Beschreibungsmorphismus ρ von $\mathfrak{P}$ nach $\mathfrak{P}'$ auf:

ρ ordnen wir die Matrix

$$\left(\begin{array}{ccc|ccc} \frac{\partial\rho(a_1)}{\partial a_1} & \cdots & \frac{\partial\rho(a_1)}{\partial a_g} & & & \\ \vdots & & \vdots & & \mathbf{0} & \\ \frac{\partial\rho(a_g)}{\partial a_1} & \cdots & \frac{\partial\rho(a_g)}{\partial a_g} & & & \\ \hline \frac{\partial\rho(r_1)}{\partial a_1} & \cdots & \frac{\partial\rho(r_1)}{\partial a_g} & \frac{\partial\rho(r_1)}{\partial r'_1} & \cdots & \frac{\partial\rho(r_1)}{\partial r'_{h'}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial\rho(r_h)}{\partial a_1} & \cdots & \frac{\partial\rho(r_h)}{\partial a_g} & \frac{\partial\rho(r_h)}{\partial r'_1} & \cdots & \frac{\partial\rho(r_h)}{\partial r'_{h'}} \end{array}\right)$$

zu, wobei wir die freien Ableitungen bereits als nach $\mathbb{Z}(H'/I')$ projeziert annehmen. Für einen Beschreibungsisomorphismus ist sie invertierbar. Da die linke obere Teilmatrix von dem Isomorphismus μ stammt, ist die Invertierbarkeit gleichbedeutend mit derjenigen der rechten unteren Teilmatrix

$$A_\rho = \begin{pmatrix} \frac{\partial \rho(r_1)}{\partial r'_1} & \cdots & \frac{\partial \rho(r_1)}{\partial r'_{h'}} \\ \vdots & & \vdots \\ \frac{\partial \rho(r_h)}{\partial r'_1} & \cdots & \frac{\partial \rho(r_h)}{\partial r'_{h'}} \end{pmatrix}$$

, wozu insbesondere

$h = h'$ vonnöten ist. Nehmen wir letzteres an, so ist A_ρ in dem Ring $\mathfrak{M}$ aller $h \times h$-Matrizen über $\mathbb{Z}(H'/I')$ enthalten. Die Matrizen, deren Zeilenvektoren die Ableitungen von Elementen aus I' nach $r'_1, \ldots, r'_h$ sind, bilden in $\mathfrak{M}$ ein Linksideal $\mathfrak{N}$. Sind ρ_1 und ρ_0 für μ zulässig, so folgt aus (17):

(35) $A_{\rho_1} - A_{\rho_0} = B \in \mathfrak{N}$.

μ gehört daher höchstens dann zu einem Beschreibungsisomorphismus, wenn

(36) $A_{\rho_0} + B$ für ein geeignetes $B \in \mathfrak{N}$ invertierbar ist.

Da die $w(a_i)$-Konjugate der $\rho_0(r_j)$ die r'_j bis auf Identitäten erzeugen, ist die von A_{ρ_0} in $\mathfrak{M} / \mathfrak{N}$ bestimmte Restklasse $\overline{A}_{\rho_0}$ stets invertierbar bezüglich Linksmultiplikation mit Elementen aus $\mathfrak{M}$, so dass wir bei den Matrizen das (27), (28) entsprechende Hochhebungsproblem haben:

(37) Kann die "Einheit" $\overline{A}_{\rho_0}$ von $\mathfrak{M} / \mathfrak{N}$ nach $\mathfrak{M}$ hochgehoben werden?

Ein Matrizenanalogon vom Typ des Schlussabsatzes von (b) ist der folgende Sachverhalt. Man betrachte die invertierbaren Modifikationen $E + B$ der Einheitsmatrix E von $\mathfrak{M}$ um Elemente aus $\mathfrak{N}$. Da $\mathfrak{N}$ ein Linksideal

ist, bilden sie eine Gruppe $\mathfrak{g}$ in $\mathfrak{M}$. Ebenso folgt daraus:

(38) Wenn A_{ρ_0} überhaupt eine invertierbare Modifikation $A_{\rho_0} + B_0$ besitzt, sind alle invertierbaren genau die Elemente der Nebenklasse $(A_{\rho_0} + B_0)\cdot \mathfrak{g}$.

Aus (36), (3) und weiteren Punkten ergibt sich der folgende Q**-Trivialitätstest:

Satz 5. Die Beschreibung $\mathfrak{P} = \{a_1, \ldots, a_g \mid R_1, \ldots, R_g\}$ zu $\pi = 1$ ist höchstens dann Q**-trivial, wenn für einen beliebigen Morphismus ρ_0 mit $\mu = \mathrm{id}_F$ in die triviale Beschreibung $\mathfrak{P}' = \{a_1, \ldots, a_g \mid R'_1 = a_1, \ldots, R'_g = a_g\}$ gilt: Die aus A_{ρ_0} durch geeignetes Verlängern hervorgehende $(g+k) \times (g+k)$-Matrix

$$\left(\begin{array}{c|cc} A_{\rho_0} & 0 & \\ \hline & 1 & 0 \\ 0 & \ddots & \\ & 0 & 1 \end{array}\right)$$

kann invertierbar gemacht werden, wenn man zu ihr eine Matrix B addiert, deren Zeilen geeignete Linearkombinationen aus Ableitungsvektoren der Peifferschen Elemente (r'_i, r'_j) nach $r'_1, \ldots, r'_{g+k}$ sind, die zu der verlängerten trivialen Beschreibung $\mathfrak{P}'_1 = \{a_1, \ldots, a_{g+k}; R'_1 = a_1, \ldots, R'_{g+k} = a_{g+k}\}$ gehören.

(d) Während bei (a), (b), (c) das Verlängern erst nachträglich in einen Isomorphietest für $\mathfrak{P}$, $\mathfrak{P}'$ eingebaut werden muss - wie bei Satz 5 geschehen -, um eine Q**-Invarianzaussage zu erhalten, ergibt sich im folgenden direkt eine solche. Wir wandeln dafür die geometrische Konsequenz, die Andrews und Curtis in [1] aus ihrer Vermutung zogen, in ein Kriterium um. Die Idee entstammt einem Gespräch mit M. M. Cohen und benutzt die Identitätentheorie nicht. (8)

(8) Analoge Sachverhalte lassen sich für den Vergleich beliebiger ϕ, ψ bei zusammenziehbaren Komplexen oder Darstellungen einer beliebigen Gruppe beweisen. Dem Vorteil, dass sich Q**-Invarianten ergeben, steht jedoch der Nachteil gegenüber, dass K und L trotz $\phi(K) = \phi(L)$ verschiedenes Einbettungsverhalten haben können, s. etwa Neuzil [9].

Satz 6. Wenn der zu einer Gruppenbeschreibung $\mathfrak{P}$ gehörige zusammenziehbare Komplex K in S^4 überhaupt nicht oder nur mit $\pi_1(S^4 - K) \neq 0$ polyedrisch eingebettet werden kann, gilt: (α) $\mathfrak{P}$ ist nicht Q**-trivial, und (β) es existiert eine 4-dim. Homotopiesphäre Σ^4, die von S^4 kombinatorisch verschieden ist.

Beweis: Jedes 2-dim. kompakte Polyeder K kann in das Innere einer geeigneten Mannigfaltigkeit M^4 semilinear eingebettet werden. N(K) sei eine reguläre Umgebung von K im Inneren von M^4. Ist K zusammenziehbar, so ist $\Sigma^4 = \partial(N \times I)$ eine Homotopiesphäre. Vermöge $N \times \{1\}$ kann K nach $\partial(N \times I)$ eingebettet werden. Sein Komplement ist dabei $N \times \{0\}$ plus ein offener Kragen um den Rand $(\partial N) \times \{0\}$, also zusammenziehbar. Insbesondere ist es einfach zusammenhängend. $N \times I$ ist reguläre Umgebung von $K \times \{\frac{1}{2}\}$. Wenn daher $\phi(K)$ die triviale Klasse ϕ_0 ist, folgt aus [1] oder [11], dass $N \times I$ eine p.l. Vollkugel D^5, also Σ^4 eine echte Sphäre ist.[9] Falls $\phi(K)$ die triviale Klasse ist, kann K also in S^4 semilinear mit $\pi_1(S^4 - K) = 0$ eingebettet werden. Die Formalumkehrung davon ist (α), und (β) folgt, da K nach der vorstehenden Überlegung dann zwar in Σ^4, aber nicht in S^4 mit einfach zusammenhängendem Komplement semilinear eingebettet werden kann, q. e. d.

4. RELATIVIERUNG

G sei eine beliebige Gruppe, $F(a_i)$ eine von $a_1, \ldots, a_g$ frei erzeugte Gruppe. In $G * F(a_i)$ betrachten wir den Normalteiler, der von $F(a_i)$ erzeugt wird. Er wird frei von den $x a_i x^{-1}$ mit $x \in G$ erzeugt und G operiert auf ihm. Äquivalent dazu können wir auch von einer freien Gruppe $F(x(a_i))$ mit den freien Erzeugenden $x(a_i)$ ($\hat{=}\ x a_i x^{-1}$) sprechen, die durch G frei permutiert werden. Unter einem G-Erzeugendensystem von $F(x(a_i))$ verstehen wir Elemente α_j derart, dass die sämtlichen $x(\alpha_j)$ $F(x(a_i))$ erzeugen; sind die $x(\alpha_j)$ eine Basis

(9) Bei [1] und [11] werden nur Q-Transformationen betrachtet. Die Invarianz regulärer 5-dim. Umgebungen bei Q**-Transformationen folgt aus den dortigen Betrachtungen jedoch unmittelbar.

im gewöhnlichen Sinn, mögen die α_j eine G-Basis heissen. Über G-Erzeugendensysteme gilt der folgende relative Nielsensche Satz, den wir am Schluss dieser Arbeit beweisen werden:

Satz 7. Ein G-Erzeugendensystem $\alpha_1, \ldots, \alpha_g$, von $F(x(a_i))$, $i = 1, \ldots, g$ der Länge $g' \leq g$ lässt sich durch eine Folge von Elementaroperationen (I) $\alpha_j \to \alpha_j^{-1}$, (II) $\alpha_j \to \alpha_j \alpha_{j'}$, $j \neq j'$, (III) $\alpha_j \to x(\alpha_j)$, $x \in G$ (die übrigen Erzeugenden bleiben jeweils ungeändert) in das System $a_1, \ldots, a_g$ überführen. Insbesondere gilt $g' = g$. (Siehe auch Punkt (a) des Anhangs.)

Die Elementaroperationen (I) und (II) erzeugen die gewöhnlichen freien Transformationen, zusammen mit (III) ergeben sich die relativen freien Transformationen.

(39) (a) Ein Übergang $\alpha_i \to \alpha_i^{\pm 1} \cdot x(\alpha_j)^{\pm 1}$ oder $\alpha_i \to x(\alpha_j)^{\pm 1} \cdot \alpha_i^{\pm 1}$, $i \neq j$ ist eine rel. freie Transformation.

(b) Permutationen sind (rel.) freie Transformationen.

Da Elementaroperationen (I), (II), (III) G-Erzeugendensysteme in G-Erzeugendensysteme und G-Basen in G-Basen überführen, ergibt sich bei Satz 7 überdies, dass mit den a_i

(40) die $\alpha_1, \ldots, \alpha_g$ eine G-Basis bilden.

Gehen wir von einem G-Automorphismus f von $F(x(a_i))$ auf sich aus, so sind die $f(a_i)$ eine G-Basis. Satz 7 ergibt dann die Aufspaltung von f in Elementarautomorphismen.

Wenn wir in $F(x(a_i))$ den von endlich vielen Elementen $R_1, \ldots, R_h$ erzeugten G-invarianten Normalteiler $N(R_j)$ bilden, erhalten wir eine relative Gruppenbeschreibung

(41) $= (1a)\ \mathfrak{P} = \{G; a_1, \ldots, a_g \,|\, R_1(x(a_i)), \ldots, R_h(x(a_i))\}$

einer Gruppe $\pi \approx F(x(a_i))/N(R_j)$. Durch $\{x(a_i) \,|\, x(R_j)\}$ ist dann eine gewöhnliche Beschreibung von π gegeben, die allerdings i. allg. nicht endlich ist. Vermöge der Operation auf $F(x(a_i))$ operiert G auf π.

Relative Q-, Q*- und Q**-Transformationen mögen analog zu den absoluten erklärt sein, nur dass statt der gewöhnlichen freien Transformationen jeweils alle relativen zugelassen sind. Die zu G gehörigen relativen (Q**-) Äquivalenzklassen bezeichnen wir mit $\phi_G(\mathfrak{P})$.

Zu G wählen wir ein (i. allg. nicht kompaktes) zusammenhängendes Polyeder L mit $\pi_1(L) \approx G$; ein solches können wir etwa erhalten, indem wir an eine Nullzelle für jedes $x \in G$ eine 1-Zelle anhängen, und zu jeder Relation unter den x eine 2-Zelle einkleben.

Einer relativen Gruppenbeschreibung $\mathfrak{P}$ zu G entspricht dann ein zusammenhängendes Polyeder $K \supseteq L$ mit endlichem K - L und Dim (K - L) ≤ 2. L ist dabei Retrakt von K bezüglich einer Retraktionsabbildung $r : K \to L$, für welche der Kern π von $r_* : \pi_1(K) \to \pi_1(L)$ gerade durch $\mathfrak{P}$ dargestellt wird. Umgekehrt lässt sich jeder Retraktionssituation[10] $r : K \to L$ für einen zusammenhängenden CW-Komplex K mit Teilkomplex L und endlichem K - L der Dim ≤ 2 eine relative Gruppenbeschreibung des r_*-Kernes mit $G = \pi_1(L)$ zuordnen, s. Cohen [4].[11] $\pi_1(K)$ ist das semidirekte Produkt aus π und G bezüglich der gegebenen Operation von G auf π.

Analog zum absoluten Fall lassen sich die zu G gehörigen relativen Beschreibungsklassen ϕ_G addieren und ergeben eine abelsche Halbgruppe $\mathfrak{K}(G)$ mit neutralem Element. Ist $f : G \to G'$ ein Homomorphismus, so induziert dieser einen Homomorphismus $f_* : \mathfrak{K}(G) \to \mathfrak{K}(G')$, indem man der Gruppenbeschreibung $\mathfrak{P} = \{G; a_1, \ldots, a_g \mid R_1(x(a_i), \ldots, R_h(x(a_i))\}$ die Beschreibung zuordnet, bei der in den Relationen die $x(a_i)$ durch die $f(x)(a_i)$ ersetzt sind.

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(10) In [7] werden etwas allgemeinere Begriffe von relativen Gruppenbeschreibungen von $\pi_1(K)$ und Elementartransformationen verwendet, die auch die Fälle erfassen, bei denen L nicht Retrakt von K ist. In dieser allgemeineren Situation lässt sich der Satz von P. Wright relativ L beweisen, und zwar sogleich für CW-Paare (K, L). Mit Hilfe von Satz 7 ergibt sich dann auch die topologische Invarianz des eingeschränkten einfachen Homotopietyps rel. L; s. Fussnote (2).

(11) Cohen versteht unter $\mathfrak{P} = \{G; a_1, \ldots, a_g \mid R_1, \ldots, R_h\}$ die Beschreibung von $\pi_1(K) \approx (F(a_i) * G)/N(R_j)$ statt des r_*-Kernes, ebenso Rothaus [15].

Im Falle der trivialen Gruppe $G = 1$ ergibt sich $\mathcal{K}(G) = \mathcal{K}$. In $\mathcal{K}(G)$ gibt es die Unterhalbgruppe, die von den Darstellungen $\mathfrak{P}$ mit $\pi = 1$ gebildet wird; und diese umfasst wiederum die Halbgruppe $\mathcal{K}^*(G)$, für die zusätzlich $g = h$ verlangt werde. $\mathfrak{P}$ bestimmt genau dann eine Klasse aus $\mathcal{K}^*(G)$, wenn bei den zugeordneten Polyedern L ein Deformationsretrakt von K ist (vergl. [4]). Vermöge $\tau(K, L)$ ergibt sich daher ein Epimorphismus $\mathcal{K}^*(G) \to Wh^*(G)$. Dabei ist $Wh^*(G)$ die Teilhalbgruppe von $Wh(G)$, die aus den Torsionswerten besteht, welche sich durch Erweiterungen K von L mit $G = \pi_1(L)$ und endlichem $K - L$ der Dimension ≤ 2 realisieren lassen, siehe [7].

Wie im absoluten Fall lassen sich $\mathcal{K}(G)$ oder Teilhalbgruppen $\mathcal{K}'(G)$ zu Gruppen komplettieren. Für induzierte Homomorphismen $f_* : \mathcal{K}'(G) \to \mathcal{K}'(G')$ resp. $\mathcal{K}^*(G) \to Wh^*(G)$ gibt es dann jeweils genau eine Fortsetzung zu Gruppenhomomorphismen.

Wir skizzieren, wie sich die Identitätentheorie auf den relativen Fall überträgt. H wird zu einer Gruppe mit den freien Erzeugenden $x(a_i)$, $x(r_j)$; $p : H \to F(x(a_i))$ werde als G-Homomorphismus durch dieselbe Formel wie im absoluten Fall erklärt. Alle weiteren Daten werden ebenfalls wie im absoluten Fall definiert mit der Massgabe, dass wir bei Untergruppen und Abbildungen durch Definition oder Beweis für Verträglichkeit mit der G-Operation sorgen. H^* ist dann z. B. eine Gruppe mit $r_1, \ldots, r_h$ als Operatorbasis bezüglich Konjugation mit Elementen aus $F(a_i) * G$; und wenn wir diese Operatoren bei Satz 7 verwenden, ergibt sich das relative Analogon zu (18). Bei (15) lautet die relative Version:

(42) = (15a) f induziert einen Isomorphismus $I/P \to H_2(\tilde{K}, \tilde{L})$,

wobei $\tilde{K}$ die universelle Überlagerung von K ist und $\tilde{L}$ der über L liegende Teilraum.

Von den sich daraus ergebenden Sätzen möge von Satz 1 exemplarisch die relative Fassung formuliert werden:

Satz 1a. Zwei Gruppenbeschreibungen $\mathfrak{P} = \{G; a_1, \ldots, a_g | R_1, \ldots, R_h$ und $\mathfrak{P}' = \{G; a_1', \ldots, a_{g'}' | R_1', \ldots, R_{h'}'\}$ zu G sind genau dann rel. äquivalent, wenn sie nach geeignetem Verlängern durch einen rel.

Morphismus ineinander abgebildet werden können, der sowohl (a) eine Homotopieäquivalenz $K \to K'$, die das gemeinsame, zu G gewählte Teilpolyeder L punktweise festlässt, als auch (b) einen Isomorphismus der relativen Peifferschen Gruppen induziert.

Auch die Ergebnisse des 3. Abschnittes lassen sich relativieren, z. B. kann man H' als Untergruppe von $F(a_i, r'_j) * G$ auffassen und einem relativen Beschreibungsmorphismus ρ wieder eine Matrix $A_\rho = \left(\frac{\partial \rho(r_j)}{\partial r'_k}\right)$ zuordnen, wobei die Glieder der Matrix aus $\mathbb{Z}((F(a_i, r'_j) * G)/I')$ sind.

Für das Wh*-Problem ergibt sich daraus folgender Test. Gehört eine rel. Beschreibung $\mathfrak{P} = \{G; a_1, \ldots, a_g \mid R_1, \ldots, R_g\}$ zu $\mathfrak{K}^*(G)$, so gibt es einen Morphismus $\rho : \mathfrak{P} \to \mathfrak{P}'$ mit $\mu = \mathrm{id}$ in die triviale Beschreibung $\mathfrak{P}' = \{G; a_1, \ldots, a_g \mid R'_1 = a_1, \ldots, R'_g = a_g\}$. Für ρ gilt (siehe die vor (37) durchgeführten Überlegungen):

(43) $\overline{A_\rho}$ ist invertierbar in $\mathfrak{M}/\mathfrak{N}$.

$\mathfrak{N}$ ergibt sich hier wegen $I' = P'$ für $\mathfrak{K}^*$ (siehe (42)) aus den Ableitungen der Peifferschen Elemente.

Vielleicht lassen sich mit Hilfe von (43) Elemente τ von $Wh(G)$ als nicht zu $Wh^*(G)$ gehörig erweisen, da $\mathbb{Z}((F(a_i, r'_j) * G)/P')$ mehr 'Nichtkommutativität' enthält als $\mathbb{Z}(G)$ und daher (43) u. U. eine echte Einschränkung im Vergleich zur Invertierbarkeit der nach $\mathbb{Z}(G)$ projezierten Matrix A_ρ bedeutet.

Während im absoluten Fall noch nicht geklärt ist, ob Q^*- und Q^{**}-Äquivalenz gleichwertig sind, wenn $\pi = 1$ und $g = h$ vorausgesetzt werden, gibt es im relativen Fall Beispiele, die die Verschiedenheit der Begriffe erweisen (vergl. [7], (34)). $G = \mathbb{Z}_m$ $(m \geq 2)$ habe das erzeugende Element x. Wir betrachten dann mit $S = a \cdot x(a) \cdot \ldots \cdot x^{m-1}(a)$ die Darstellung

(44) $\mathfrak{P} = \{\mathbb{Z}_m; a \mid R = S \cdot x(a) \cdot x(S)^{-1}\}$.

Sie ist nicht rel. Q^*-trivial, da das von R in $\mathbb{Z} * \mathbb{Z}_m$ bestimmte zyklisch gekürzte Wort von $a^{\pm 1}$ verschieden ist. Die zugehörige Gruppe

ist $\pi = 1$, da $R \cdot x(R) \cdot \ldots \cdot x^{m-1}(R)$ ein Konjugat von $x(S)$ ist, also S und somit auch a zu $N(R)$ gehört. Wenn wir $\mathfrak{P}$ um eine weitere Erzeugende b zu $\mathfrak{P}_1$ verlängern, ergibt die folgende Q-Transformationsfolge bei den Relationen die Q-Trivialität von $\mathfrak{P}_1$:
$R,\ b \to R,\ S \cdot b \to b^{-1} \cdot x(a) \cdot x(b),\ S \cdot b \overset{*}{\to} b^{-1} \cdot x(a) \cdot x(b),\ b \to a,\ b.$
Dabei gilt der mit $*$ bezeichnete Übergang, weil das Produkt der x^i-Konjugate, $i = 0, \ldots, m-1$ von $b^{-1} \cdot x(a) \cdot x(b)$ wiederum ein Konjugat von $x(S)$ ergibt. Siehe auch Punkt (e) des Anhangs.

Die Beispiele von (α) Osborne [11] und (β) Neumann-Rapaport [13], die vielleicht die Andrews-Curtis Vermutung widerlegen, entstehen übrigens auch aus relativen Beschreibungen durch Vernachlässigung der G-Aktion:

(45) = (31a) (α) $\mathfrak{P} = \{\mathbb{Z}_2; a \mid a^{-3} \cdot x(a)^{-1} \cdot a^2 \cdot x(a)\}$ resp.
(β) $\mathfrak{P} = \{\mathbb{Z}_3; a \mid x(a)^{-2} \cdot a^{-1} \cdot x(a) \cdot a\}$. Dabei ist x erzeugendes Element von $\mathbb{Z}_2$ resp. $\mathbb{Z}_3$.

Satz 8. $f : K \to K'$ sei eine Abbildung zwischen zusammenhängenden CW-Komplexen, die den gemeinsamen zusammenhängenden[12] Teilkomplex L punktweise festlässt und einen Isomorphismus $\pi_1(K) \to \pi_1(K')$ induziert. Dabei seien $K - L$ und $K' - L$ endlich mit $\mathrm{Dim}(K' - L) \leq 1$. Dann ist f eine einfache Homotopieäquivalenz und homotop rel. L zu einer formalen Deformation $g : K \to K'$, bei der höchstens 2-dimensionale Erweiterungen nötig sind.

Beweis: $\rho : K_1 \to K$ und $\rho' : K' \to K_1'$ seien formale Deformationen rel. L der Dimension ≤ 2 derart, dass K_1 und K_1' aus L durch Anhängen je einer endlichen Anzahl von Schleifen an eine Nullzelle e^0 hervorgehen. $a_1, \ldots, a_g$ resp. $a_1', \ldots, a_{g'}'$ seien durch sie bestimmte Erzeugende von $\pi_1(K_1') \approx \pi_1(L) * F(a_i')$, wobei e^0 als Basispunkt dienen möge. Bei $K_1 \overset{\rho}{\to} K \overset{f}{\to} K' \overset{\rho'}{\to} K_1'$ können wir ferner o. B. d. A. annehmen, dass a_i in ein Produkt der $x a_i' x^{-1}$, $x \in \pi_1(L)$ und ihrer

(12) Für zusammenhängendes L ergibt sich diese Antwort auf die Frage von Wall unmittelbar aus Satz 7. Ich habe nicht geprüft, ob die Verallgemeinerung auf Schwierigkeiten stösst, falls nur K und K' zusammenhängend sind. Vergl. Anhang, (a).

Inversen übergeht, denn durch Vorschalten einer weiteren formalen Deformation rel. L der Dim ≤ 2 vor ρ können wir dies sonst immer erreichen. Der durch $\rho' f \rho$ induzierte Isomorphismus $\pi_1(L) * F(a_i) \to \pi_1(L) * F(a_i')$ bildet daher nicht nur $\pi_1(L)$ identisch auf sich ab, sondern es entsteht auch ein mit der $\pi_1(L)$-Operation verträglicher Isomorphismus $\psi : F(x(a_i)) \to F(x(a_i'))$. Aus Satz 7 folgt nun $g = g'$ und die Existenz einer Folge von Elementarschritten (I), (II), (III), welche die Operatorbasis $\psi(a_i)$, $i = 1, \ldots, g$ in die aus den a_i' bestehende überführt. Diese Folge ergibt, dass $\rho' f \rho$ homotop rel. L ist zu einer formalen Deformation $g_1 : K_1 \to K_1'$ der Dim ≤ 2. Vermöge der zu ρ resp. ρ' inversen formalen Deformationen $\bar{\rho} : K \to K_1$ resp. $\bar{\rho}' : K_1' \to K'$ folgt dann: $f \simeq g = \bar{\rho}' g_1 \bar{\rho}$, und g hat mit seinen Faktoren die gewünschten Eigenschaften, q. e. d.

Wir haben noch den Beweis des rel. Nielsenschen Satzes nachzutragen. Er wird in Anlehnung an Nielsen [10] geführt, wobei wir ihn zur Abwechslung in einen indirekten Beweis verwandeln. Die Idee besteht darin, dass man auf die $x(\alpha_j)$ als Worte in den $x(a_i)$ die gewöhnliche Kürzungsmethode anwendet und zu erreichen versucht, dass dies G-äquivariant möglich ist. Dabei stösst man auf ein Hindernis, wenn z. B. ein α_j bei der gewöhnlichen Kürzungsmethode mit $x(\alpha_j)$ multipliziert wird. Die Betrachtung (47) zu Beweisbeginn dient daher dem Nachweis, dass ein solches Hindernis in Wirklichkeit unter der Voraussetzung $g' \leq g$ nicht auftritt.

Beweis von Satz 7: Der Epimorphismus $p : F(x(a_i)) \to F(a_i)$ entstehe durch 'Herauskürzen der G-Operation', d. h., jedes $x(a_i)$ werde auf a_i abgebildet. Da $F(a_i)$ eine freie Gruppe vom Rang g ist und von den $p(\alpha_j)$ erzeugt wird, ergibt sich

(46) $g' \geq g$ und zusammen mit der Voraussetzung $g' \leq g$ die Schlussbehauptung.

Für ein G-Erzeugendensystem $\alpha_1, \ldots, \alpha_g$ folgt, dass sich

(47) α_j und $x(\alpha_j)$ bei $\alpha_j \cdot x(\alpha_j)$ stets um weniger als die Hälfte der Länge $l(\alpha_j)$ kürzen, [13]

denn wenn α_j gerade Länge hat, müsste es sonst die Gestalt $\alpha_j = r \cdot x(r)^{-1}$ haben, was zur Folge hätte, dass $F(a_i)$ wegen $p(\alpha_j) = 1$ bereits von den übrigen $p(\alpha_i)$ erzeugt wird, im Widerspruch zu $\mathrm{Rang}(F(a_i)) = g$. Im Falle ungerader Länge von α_j müsste sich sonst das mittlere Erzeugendensymbol von α_j gegen das mittlere von $x(\alpha_j)$ kürzen, was wegen der Gleichheit der Exponenten unmöglich ist.

$\alpha_1', \ldots, \alpha_g'$ sei ein System, dass sich aus den α_i durch endliche Anwendung der Elementaroperationen (I), (II), (III) ergibt und minimale Längensumme $\sum_{i=1}^{g} l(\alpha_i')$ hat. Wir zeigen durch Herbeiführung eines Widerspruchs, dass dann kein α_i' eine Länge ≥ 2 hat. Dann folgt aus Betrachtungen analog zu (46), dass die α_i' eine Permutation der $x_1(a_1)^{\varepsilon_1}, \ldots, x_g(a_g)^{\varepsilon_g}$, $\varepsilon_i = \pm 1$ sind und mit weiteren freien Transformationen daher in die Basis aus den a_i überführt werden können (s. (39b)).

Gäbe es ein α_i' mit $l(\alpha_i') \geq 2$, so existierte gemäss Nielsen [10] ein Kurzwort w_0 in den $x(\alpha_i')$ mit $a_{i_0} = w_0(x(\alpha_i'))$ in $F(x(a_i))$, bei dem entweder

(*) ein $x_1(\alpha_{i_1}')^{\pm 1}$ von einem Nachbarelement $x_2(\alpha_{i_2}')^{\pm 1}$ in w_0 um mehr als die Hälfte seiner Länge gekürzt wird oder

(**) ein $x_1(\alpha_{i_1}')^{\pm 1}$ von gerader Länge ist und von seinen Nachbarelementen in w_0 je zur Hälfte gekürzt wird.

Der Fall (*) kann jedoch in Wirklichkeit nicht eintreten, denn für $i_1 = i_2$ verbietet ihn (47) bei gleichen Exponenten; bei ungleichen Exponenten müsste $x_1 = x_2$ sein, und w_0 wäre nicht kurz gewesen. Für $i_1 \neq i_2$ ergäbe sich die Möglichkeit einer Transformation, die die Gesamtlänge weiter reduziert, im Widerspruch zu deren vorausgesetzter Minimalität.

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(13) Unter der Länge eines Elementes von $F(x(a_i))$ verstehen wir die gewöhnliche Länge des Kurzwortes in den $x(a_i)$.

Im Fall (**) ergibt sich jetzt zusätzlich, dass die betreffenden Nachbarelemente mindestens die Länge von α'_{i_1} haben, denn sonst würde für ein solches Nachbarelement doch die Situation (*) vorliegen. Wenn wir von $x_1(\alpha'_{i_1})^{\pm 1}$ und einem geeigneten seiner beiden Nachbarelemente in w_0 ausgehen, erhalten wir aus (**) daher die Existenz einer Situation

(48) $\alpha'_{i_1} = u \cdot v$, $x(\alpha'_{i_2})^{\varepsilon} = v^{-1} \cdot w$, $\varepsilon = \pm 1$, u, v, w und $u \cdot v$, $v^{-1} \cdot w$ kurz, $l(u) = l(v) \leq l(w)$ und $i_1 \neq i_2$,

wobei $i_1 \neq i_2$ wiederum aus (47) und der Kürze von w_0 folgt.

Wir zeigen jetzt, dass man auf die α'_i weitere rel. freie Transformationen anwenden kann, die die Länge jedes Elementes nicht ändern - (*) bleibt also weiterhin ausgeschlossen -, und zwar so, dass am Ende auch (48) und damit (**) nicht mehr auftritt (2. Nielsenscher Prozess). Der Endzustand ergibt dann den gewünschten Widerspruch.

Wegen (39b) können wir o. B. d. A. $l(\alpha'_1) \leq l(\alpha'_2) \leq \ldots \leq l(\alpha'_g)$ annehmen. α'_{i_1} sei das früheste Element in dieser Anordnung der α'_i mit der Eigenschaft (48). Wenn dann ein α'_{i_2} mit $i_2 \neq i_1$ (von mindestens gleicher Länge wie α'_{i_1}) mit $x^{-1}(v)^{-1}$ anfängt, ersetzen wir diesen Anfangsteil durch $x^{-1}(u)$. Wenn α'_{i_2} mit $x^{-1}(v)$ endet, ersetzen wir dies Endstück durch $x^{-1}(u)^{-1}$. Beide Prozesse sind vom Typ (39a), also rel. freie Transformationen. Sie erhalten die Länge von α'_{i_2}, da sonst $\sum_{i=1}^{g} l(\alpha'_i)$ verkleinert werden könnte. Insbesondere kommen sich daher der Anfangs- und Endstückaustausch nie 'in die Quere', obwohl es möglich ist, dass beide auf ein α'_{i_2} angewendet werden müssen.

(49) Nach diesen Austauschprozessen hat α'_{i_1} die Eigenschaft (48) nicht mehr,

denn die Notwendigkeit eines weiteren Austausches müsste sich auf ein Anfangs- oder Endstück beziehen, das schon einmal ausgetauscht wurde; dann ergäbe sich jedoch $x^{-1}(u) = y^{-1}(v)^{-1}$ und somit $p(\alpha'_{i_1}) = 1$, was bei der Begründung von (47) bereits als unmöglich erwiesen wurde.

Wenn wir das Erzeugendensystem nach dem Austauschen mit α_i'' bezeichnen, gilt weiter:

(50) Kein Element α_{i_2}'' vor α_{i_1}'' $(= \alpha_{i_1}')$ hat die Eigenschaft (48) neu erworben,

denn ein solches Element α_{i_2}'' hat höchstens die Länge von α_{i_1}'. Ist sie kleiner als die von α_{i_1}', gilt: $\alpha_{i_2}'' = \alpha_{i_2}'$; ist sie gleich der von α_{i_1}', kann höchstens das Anfangsstück von α_{i_2}' ausgetauscht worden sein, denn sonst hätte bereits α_{i_2}' die Eigenschaft (48) gehabt. In jedem Fall hat also ein α_{i_2}'' von gerader Länge vor α_{i_1}'' dieselbe rechte Hälfte v' wie α_{i_2}' mit $l(v') \leq l(u) = l(v)$. v'^{-1} war vor den Austauschprozessen nicht Anfang eines $x(\alpha_i')^{\varepsilon}$ mit $i \neq i_2$, $l(\alpha_i') \geq l(\alpha_{i_2}')$, insbesondere nicht für $i = i_1$, d. h. v'^{-1} ist nicht Anfang eines $x(u)$ oder $x(v)^{-1}$. Die $x(\alpha_i'')$ mit $i \neq i_2$, $l(\alpha_i'') \geq l(\alpha_{i_2}'')$ haben als Anfangsstücke der Länge $l(v') \leq l(u)$ dieselben Gesamtmöglichkeiten wie zuvor: diese wurden nämlich entweder beibehalten oder als ein Anfangsstück eines $x^{-1}(v)^{-1}$ gegen das gleicher Länge von $x^{-1}(u)$ ersetzt, die aber beide wegen $\alpha_{i_1}'' = \alpha_{i_1}'$ ohnehin vertreten waren und es weiter sind.

v'^{-1} ist also auch nicht Anfang eines $x(\alpha_i'')$ mit $i \neq i_2$, $l(\alpha_i'') \geq l(\alpha_{i_2}'')$, d. h. (50) ist nachgewiesen.

Aus (49) und (50) folgt nun, dass in dem Erzeugendensystem $\alpha_1'', \ldots, \alpha_g''$ mit $l(\alpha_1'') \leq l(\alpha_2'') \leq \ldots \leq l(\alpha_g'')$ das erste Element mit der Eigenschaft (48) einen grösseren Index als i_1 hat. Wir können daher induktiv den Fall (48) überhaupt ausschliessen, q. e. d.

ANHANG

Ich möchte noch eine Übersicht über einige Resultate geben, die im Zusammenhang mit der vorliegenden Arbeit entstanden sind. In den meisten Fällen sind die Beweise zu umfangreich und sollen daher an anderer Stelle publiziert werden.

(a) Zusammen mit Gert Denk habe ich einen relativen Grushko-Neumann-Satz erhalten. In $G * G_1 * \ldots * G_m$ betrachte man den von den G_i erzeugten Normalteiler F. Dann gilt mit analogen Begriffen wie zu Beginn von Abschnitt 4 der

Satz. Ein G-Erzeugendensystem $\alpha_1, \ldots, \alpha_n$ von F lässt sich durch relative freie Transformationen in ein solches überführen, bei dem jede Erzeugende in einem G_i liegt.

Da $n \neq m$ zugelassen ist, kann daraus insbesondere die Voraussetzung $g' \leq g$ bei dem rel. Nielsenschen Satz 7 als entbehrlich erwiesen werden; im Falle $g' > g$ entstehen dabei $g' - g$ triviale Erzeugende.

Unser Beweis ist eine Variante der üblichen Kürzungsmethode. Wir haben jedoch vor, zu untersuchen, ob auch die geometrische Methode von Stallings zum Ziel führt. Weiter ergeben sich vermutlich - wie im absoluten Fall - Versionen des Satzes für Gruppoide bzw. für beliebige Indexmengen. (Die Gruppoidversion ist evtl. für Fussnote (12) hilfreich.) Letztens: Auch andere Sätze der kombinatorischen Gruppentheorie - z. B. den Satz von Kurosch - wollen wir daraufhin untersuchen, ob sie ein Analogon im relativen Fall besitzen.

(b) Bezüglich der in 1. (a) II genannten Fragen habe ich kürzlich das folgende Resultat erzielt:

Aus $\phi(K) = \phi(L)$ folgt $K' \times I \searrow L$, wobei K' ein zweidimensionales Polyeder mit $K \nearrow K'$ ist.

Als Spezialfall für $\phi(K) = \phi_0$ zu $\pi = 1$ ergibt sich daher $K' \times I \searrow 0$ für ein aus K durch ≤ 2-dimensionale Erweiterungen hervorgehendes Polyeder K'. Gegenwärtig untersuche ich, ob diese zweidimensionalen Erweiterungen rückgängig gemacht werden können, ohne dass die 'Zeeman-Eigenschaft' verloren geht.

Für gewisse induktiv konstruierte Serien zusammenziehbarer Komplexe K^2 mit $\phi(K^2) = \phi_0$ haben Winfried Becker und Albert Zimmermann die Zeemansche Vermutung bewiesen.

(c) Aus einem Resultat von Klaus Sauermann und Gabriele Wessel folgt, dass zu den Beschreibungen $\{a_1, \ldots, a_g \mid R_1, \ldots, R_g\}$, bei denen die R_j freie Transformierte der a_i sind, gewisse 'modifizierte' Standardkomplexe K^2 gehören, die $K^2 \times I \searrow 0$ erfüllen. Diese kollabieren sogar prismatisch, d. h. das 3-dim. 'Material' kann vollständig entfernt werden, indem man für jede 2-Zelle e_i^2 von K^2 $\bar{e}_i^2 \times I$ von oben, unten oder beiden Richtungen bis auf je eine schräg liegende 2-Zelle kollabiert und so einen kollabierbaren zweidimensionalen Rest erhält.

Marshall M. Cohen hat hierzu kürzlich die Umkehrung bewiesen:

(51) Wenn $K^2 \times I$ prismatisch kollabiert, bestimmen die 2-Zellen von K^2 Relationen R_j, von denen geeignete Konjugierte $w_j R_j w_j^{-1}$ freie Transformierte der Erzeugenden sind.

Da sich die technischen Einschränkungen an die Komplexe, die in dem Verfahren von Sauermann und Wessel bisher nötig sind, evtl. beseitigen lassen, ist prismatische Kollabierbarkeit vermutlich durch (51) charakterisiert.

Die ersten Untersuchungen über prismatische Kollabierbarkeit stammen von A. Zimmermann, der in seiner Dissertation ein notwendiges Kriterium für sie angegeben hat, welches die Umgebungsgraphen (= link) der Eckpunkte von K^2 betrifft. Damit konnte er z. B. zeigen, dass $K^2 \times I$ bezüglich keiner semilinearen Zellaufteilung von $|K^2|$ prismatisch kollabiert, wenn K^2 der Standardkomplex zu $\{a, b \mid a^2 b^3, ab\}$ ist.

(d) Die folgende Beobachtung bezieht sich ebenfalls auf 1. (a) II:

(52) Für ein nichttriviales aber invertierbares $\phi(K^2)$ mit $\phi(K^2) + \phi(L^2) = \phi_0$ kollabiert $(K \vee L) \times I$ nicht, obwohl $\phi(K \vee L) = \phi_0$ erfüllt ist.

Wenn nämlich $(K \vee L) \times I \searrow 0$ gilt, so kollabieren $K \times I$ und $L \times I$ auf eindimensionale Teilpolyeder und damit überhaupt, da letztere Bäume sein müssen. $K \times I \searrow 0$ ist aber mit $\phi(K) \neq \phi_0$ unverträglich. Man kann dies

Ergebnis auch so formulieren:

(53) Falls die Vermutung von Zeeman für die triviale Klasse gilt, ist ein nichttriviales $\phi \in \mathfrak{K}^*$ nicht invertierbar (vergl. die Situation bei 3-Mannigfaltigkeiten bezüglich #).

(e) Die für (44) verwendete Konstruktionsidee liefert auch das folgende Beispiel:

(54) $\mathfrak{P} = \{\mathbb{Z}_2 * \mathbb{Z};\ a \mid R = hg(a \cdot h(a)) \cdot a \cdot g(a \cdot h(a))^{-1}\}$, wobei h, g $\mathbb{Z}_2$ resp. $\mathbb{Z}$ erzeugen.

Da $R \cdot h(R)$ ein Konjugat von $a \cdot h(a)$ ist, gilt $\pi = 1$. Bei den zugehörigen Komplexen $K \supset L$ ist die Berandungsabbildung $C_2(\tilde{K}, \tilde{L}) \to C_1(\tilde{K}, \tilde{L})$ durch die nichttriviale Einheit[14]

(55) $u = hg + h\,g\,h + 1 - g - g\,h \in \mathbb{Z}(G),\ G = \mathbb{Z}_2 * \mathbb{Z}$

gegeben. Sie bestimmt jedoch kein nichttriviales Element aus $Wh^*(G)$, denn (α) ist $Wh(G) = 0$ und (β) lässt sich nach dem im Anschluss an (44) beschriebenen Verfahren für alle zu (54) analogen Beispiele mit $\mathbb{Z}_m$ (statt $\mathbb{Z}_2$) die Q**-Trivialität feststellen, obwohl $Wh(G)$ im allgemeinen nicht verschwindet. Trotzdem ist (54) hier wiedergegeben, denn u. U. ergibt eine kleine Modifikation nicht nur nichttriviale Einheiten, sondern sogar Elemente, die ein $Wh^*(G)$ als von Null verschieden erweisen.

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18 · Two-dimensional complexes with torsion values not realizable by self-equivalences*

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1. STATEMENT OF PROBLEM AND DISCUSSION OF RESULT

The study of the group $\varepsilon(K^2)$ of self-equivalences of a two-dimensional complex K^2 has so far led only to cases, where all values of the Whitehead group can be realized as $\tau(f)$, $[f] \in \varepsilon(K^2)$, see Cockroft and Moss [2], Dyer and Sieradski [5] and Olum [8]. It is the aim of the present paper to show by examples that this is not true in general; there exist nonrealizable torsion values even for finitely generated fundamental groups:

Theorem 1. *The standard complex K^2 of the presentation $\{a, b \mid b^p, a\,b\,a^{-1}b^{-1}\}$ of $\pi = \mathbb{Z} \times \mathbb{Z}_p$, p prime, has nonrealizable torsion values if and only if the class number, h(p), of the p-th cyclotomic field is different from 1.* (1)

This result has consequences for the problem, which torsion values lie in $Wh^*(\pi)$, (2) and the still unsolved question (see Cohen [3], p. 81 and problem D6 of this volume), whether homotopy type equals simple-homotopy type for all finite 2-complexes:

Theorem 2. *If τ_0 is nonrealizable with respect to $\varepsilon(K^2)$, but $-\tau_0 \in Wh^*(\pi_1(K^2))$, then the corresponding extension $L^2 \supset K^2$ gives rise to complexes L and K with the same homotopy type but different simple-homotopy types.*

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* With an appendix on algebraic K-theory by C. T. C. Wall.

(1) In Theorem 1' below we will state in detail which torsion values can be realized. The smallest prime with $h(p) > 1$ is known to be $p = 23$ with $h = 3$.

(2) that is, which occur as $\tau(L, K)$ for a finite CW-pair with $L \searrow K$, $\dim(L - K) \leq 2$ and $\pi = \pi_1(K)$. See [4], [6], [7], [9] and problem D8.

Thus Theorem 1 gives either

(A) values $-\tau_0 \notin Wh^*$ - like Rothaus [9] -

or examples showing that

(B) even in dimension 2 simple-homotopy type is a finer classification than homotopy type.

Which of these alternatives (A), (B) really holds, or if both can occur, possibly even in one example, is a topic for further research.

Proof of Theorem 2. If $\tau(L, K) = -\tau_0 \in Wh^*$, then the deformation retraction r of $K \overset{i}{\hookrightarrow} L \overset{r}{\to} K$ fulfils $\tau(r) = \tau_0$, since $0 = \tau(ri) = \tau(r) + r_*\tau(i) = \tau(r) + r_*i_*\tau(L, K) = \tau(r) + \tau(L, K)$ (see Cohen [3], p. 72).

An additional simple-homotopy equivalence, $f : K \to L$, would yield the self-equivalence rf of K having the 'forbidden' value $\tau(rf) = \tau(r) + r_*\tau(f) = \tau_0 + r_*(0) = \tau_0$, q. e. d.

With τ_0 nonrealizable with respect to $\varepsilon(K^2)$, the same conclusion even implies that an arbitrary complex L, homotopy equivalent to K^2 by a map $L \to K^2$ with torsion τ_0, fulfils $L \not\simeq K^2$.

As there always exist deformation retractions $L^3 \rightsquigarrow K^2$ with prescribed τ_0, the question arises whether

L^3 has the simple-homotopy type of a 2-dimensional complex, if τ_0 is nonrealizable with respect to $\varepsilon(K^2)$.

Compare this with the discussion of Cockroft and Moss [2], as well as Cohen [3] from p. 79 on. This discussion is continued by the present paper.

Before starting with details I would like to make a comment on the origin of this paper.

(1) Let f, f' be continuous maps $K^2 \to K^2$, which coincide on the 1-skeleton. Then the images of the 2-cells of a cellular structure lifted to the universal covering $\tilde{K}^2$ differ by elements of $H_2(\tilde{K}^2)$.

Therefore, it is appropriate to look for complexes with

'small' $H_2(\tilde{K}^2)$ (and 'few' π_1-automorphisms)

but

'large' $\mathrm{Wh}(\pi_1)$,

in order to obtain nonrealizable torsion values. This is suggested too by the fact that an element of $\mathrm{Wh}(\pi_1)$ always becomes realizable if one takes the wedge product of K^2 with a finite number of 2-spheres.

At first I thought of one-relator groups in this context. If $\mathrm{Wh}(\pi)$ of a one-relator group contains elements which cannot be represented by 1×1 matrices, then at least they cannot be realized by self-equivalences f of the standard complex K^2 which induce the identity isomorphism f_* of $\pi = \pi_1$. But as far as I know such elements of the Whitehead-group of one-relator groups are not known (compare problem A0).

I had already tested several finite groups in vain, when at the Durham symposium C. T. C. Wall and H. Bass pointed out to me the result about the algebraic K-theory of $\mathbb{Z} \times \mathbb{Z}_p$ which is cited in (12). I am very much indebted to them, because their explanations were the missing link between what I knew and what I wanted to obtain. Without 'Durham' this paper probably would not have come into existence.

2. CONSTRUCTION AND PROOFS

(a) Consider the group $\pi = \mathbb{Z} \times \mathbb{Z}_m$, $m \in \mathbb{N}$, and the standard complex K^2 defined by its presentation $\{a, b \mid b^m, a\,b\,a^{-1}b^{-1}\}$; (in (d) we will specialize to the case of a prime $m = p$). With respect to an appropriate fundamental system $\tilde{a}$, $\tilde{b}$, $\tilde{R}_1$, $\tilde{R}_2$ of 1- and 2-dimensional cells the boundary relations of $\tilde{K}$ are the Fox derivatives of the defining relators:

$$(2) \qquad \partial\tilde{R}_1 = 0\cdot\tilde{a} + \Big(\sum_{i=0}^{m-1} b^i\Big)\cdot\tilde{b}$$

$$\partial\tilde{R}_2 = (1 - b)\cdot\tilde{a} + (a - 1)\cdot\tilde{b}\,.$$

As can be seen immediately, this implies that the 2-chains $Z_1 = (1 - b)\cdot\tilde{R}_1$ and $Z_2 = (a - 1)\cdot\tilde{R}_1 - \sum\cdot\tilde{R}_2$ are cycles, with $\sum$ as an abbreviation for

$\sum_{i=0}^{m-1} b^i$. But, moreover

(3) Z_1 and Z_2 generate $H_2(\tilde{K})$,

since an arbitrary chain $c_2 \in C_2(\tilde{K})$ can be modified to $c_2' = n \cdot \tilde{R}_1 + \beta \cdot \tilde{R}_2$, $n \in \mathbb{Z}$, $\beta \in \mathbb{Z}(\pi)$, by a $\mathbb{Z}(\pi)$-linear combination of Z_1 and Z_2, having boundary

$$\partial c_2' = \beta \cdot (1 - b) \cdot \tilde{a} + (\beta \cdot (a - 1) + n \cdot \sum) \cdot \tilde{b}.$$

In the case of a cycle c_2 we end up with a cycle c_2'. In particular we get $\beta \cdot (a - 1) + n \cdot \sum = 0$. Passing over to coefficient sums gives $n \cdot m = 0$. Thus n vanishes.

Because of $0 = \beta \cdot (a - 1) + n \cdot \sum = \beta \cdot (a - 1)$, β must vanish too, as $a - 1$ is not a zero divisor in $\mathbb{Z}(\pi)$. We have shown $c_2' = 0$ for a cycle c_2, so c_2 is a linear combination of Z_1 and Z_2. (Remark that Z_1 and Z_2 are linearly dependent.)

(b) Obviously the possible π_1-automorphisms are given by

$a \to a^{\varepsilon} \cdot b^x$
$b \to b^r$ with integer exponents $\varepsilon = \pm 1$,

$-1 \leq r \leq m - 2$, $(r, m) = 1$. Next we show that

(4) at most those with $r = \pm 1$ can be induced by a homotopy equivalence $K \to K$.

(Lemma 1 and Theorem 1' below will show that this is even a sufficient condition.)

Proof of (4). By the homotopy extension property for CW-complexes we may assume without loss of generality that a self-equivalence $f: K \to K$ is normalized in the 1-skeleton such that $\tilde{f}_1$ of the diagram

(5)
$$\begin{array}{ccc} C_2(\tilde{K}) & \xrightarrow{\tilde{f}_2} & C_2(\tilde{K}) \\ \downarrow \partial & & \downarrow \partial \\ C_1(\tilde{K}) & \xrightarrow{\tilde{f}_1} & C_1(\tilde{K}) \end{array}$$

is given by the matrix

$$(6)\qquad \begin{pmatrix} \varepsilon \cdot a^{(\varepsilon-1)/2}, & a^{\varepsilon} \cdot \dfrac{b^{x}-1}{b-1} \\ 0, & \dfrac{b^{r}-1}{b-1} \end{pmatrix}$$

of Fox derivatives of the π_1-automorphism.

(Here, $\frac{b^n-1}{b-1}$ is an abbreviation for the corresponding geometric series, in the case where $n < 0$: $-b^{-1} - b^{-2} - \ldots - b^{n}$.) In what follows we will always assume this normalization to be given.

If $\tilde{f}_2$ has matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, using (2), (5) and (6) we arrive at the matrix equality:

$$(7)\qquad \begin{pmatrix} 0, & \Sigma \\ 1-b^{r}, & a^{\varepsilon} \cdot b^{x}-1 \end{pmatrix} \cdot \begin{pmatrix} \varepsilon \cdot a^{(\varepsilon-1)/2}, & a^{\varepsilon} \cdot \dfrac{b^{x}-1}{b-1} \\ 0, & \dfrac{b^{r}-1}{b-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0, & \Sigma \\ 1-b, & a-1 \end{pmatrix} .$$

Comparing the left lower entries of the resulting matrices on both sides, we get

$$(1 - b^{r}) \cdot \varepsilon \cdot a^{(\varepsilon-1)/2} = \delta \cdot (1 - b).$$

First mapping $a \to 1$ and then arguing with congruences as in [6], p. 17, this gives

$$(8)\qquad \delta^{0} \equiv r \cdot \varepsilon \mod m$$

as a value for the coefficient sum δ^0.

Now (2) implies immediately that $H_2(K)$ is a free group of rank 1 freely generated by R_2 (the 2-cell of K, which is covered by $\tilde{R}_2$). Moreover δ^0 defines the $H_2(K)$-homomorphism. Thus in the case of a self-equivalence we have $\delta^0 = \pm 1$, and by (8) and the fact that $-1 \le r \le m-2$, (4) is proved.

(c) We now look for solutions $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of the equation (7) with the property $r = \pm 1$. It is well known that they can be realized geometrically. A particular solution is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} r\,, & 0 \\ 0\,, & \varepsilon \cdot r \cdot a^{(\varepsilon-1)/2} \cdot b^{(r-1)/2} \end{pmatrix}$$

as can be seen by direct computation.

Thus (1) and (3) give the general solution

$$(9) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} r + \lambda_1(1-b) + \mu_1(a-1), & -\mu_1 \cdot \Sigma \\ \lambda_2(1-b) + \mu_2(a-1) \quad , & \varepsilon \cdot r \cdot a^{(\varepsilon-1)/2} \cdot b^{(r-1)/2} - \mu_2 \cdot \Sigma \end{pmatrix},$$

$$\lambda_i,\ \mu_i \in \mathbb{Z}(\pi)\ .$$

$\tilde{f}_0$ is always an isomorphism and by (6), so is $\tilde{f}_1$ in the case where $r = \pm 1$. Hence we have

(10) f is a self-equivalence if and only if the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in (9) describing $\tilde{f}_2$ is invertible.

The matrix of $\tilde{f}_0$ $(= \mathrm{id}_{C_0(\tilde{K})})$ moreover belongs to the trivial Whitehead class, and on account of (6) the same holds for that of $\tilde{f}_1$ in the case $r = \pm 1$. Thus we have for a (normalized) self-equivalence:

$$(11) \qquad \tau(f) = \tau\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad .$$

Setting $\lambda_2 = \mu_2 = 0$ and varying λ_1, μ_1, we get by (9), (10) and (11):

Lemma 1. In the case $r = \pm 1$, all torsion values can be realized by self-equivalences of K^2, which are totally in the left summand of the decomposition

$$\mathrm{Wh}(\pi) = (U(\mathbb{Z}(\pi))/\pm\pi) \oplus SK_1(\mathbb{Z}(\pi)).$$

In fact, a unit of $\mathbb{Z}(\pi)$ can be written as $r + \lambda_1 \cdot (1-b) + \mu_1 \cdot (a-1)$ up to sign.

We want to prove that the condition of Lemma 1 is also necessary in the case of a prime $m = p$. The key observation is the following: An invertible matrix of type (9) becomes elementary equivalent to a 1×1-

matrix if we factor the group ring by the ideal $(\sum)$, because then it takes the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Thus:

Lemma 2. *Let $\Lambda = \mathbb{Z}(\mathbb{Z} \times \mathbb{Z}_m)/(\sum)$. Then the $\tilde{f}_2$-matrix, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, of a normalized self-equivalence of K^2 determines an element of the left summand in the decomposition $K_1(\Lambda) = U(\Lambda) \oplus SK_1(\Lambda)$.*

(d) Because of $\pi = \mathbb{Z} \times \mathbb{Z}_m$, $\mathbb{Z}(\pi)$ can be viewed as the ring of L-polynomials over $\mathbb{Z}(\mathbb{Z}_m)$. When m is a prime p, to which case we will restrict ourselves from now on, $\Lambda = \mathbb{Z}(\pi)/(\sum)$ is the ring of L-polynomials over $\mathbb{Z}(\mathbb{Z}_p)/(\sum)$, the integers of the p-th cyclotomic field K_p. We use now (see the appendix for an outline of the proof):

(12) (i) The projection $SK_1(\mathbb{Z}(\pi)) \to SK_1(\Lambda)$ is an isomorphism.
(ii) Both groups in (i) are isomorphic to the ideal class group of K_p. Thus they have order $h(p)$.

Now look at the diagram

$$\begin{array}{ccccc} Wh(\pi) & = & (U(\mathbb{Z}(\pi))/\pm\pi) & \oplus & SK_1(\mathbb{Z}(\pi)) \\ \uparrow & & \uparrow & & \uparrow \approx \\ K_1(\mathbb{Z}(\pi)) & = & U(\mathbb{Z}(\pi)) & \oplus & SK_1(\mathbb{Z}(\pi)) \\ \downarrow & & \downarrow & & \downarrow \approx \text{ by (12i)} \\ K_1(\Lambda) & = & U(\Lambda) & \oplus & SK_1(\Lambda) \end{array} .$$

It reveals that a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ having a nontrivial SK_1-component with respect to $Wh(\pi)$ also gives rise to a nontrivial SK_1-part in $K_1(\Lambda)$. However, for the $\tilde{f}_2$-matrix of a normalized self-equivalence of K^2 the latter cannot occur, as lemma 2 shows. Together with Lemma 1 this consideration implies:

Theorem 1'. *Exactly those torsion values τ_0 can be realized by self-equivalences of K^2, which lie in the left summand of the decomposition*

$$Wh(\pi) = (U(\mathbb{Z}(\pi))/\pm\pi) \oplus SK_1(\mathbb{Z}(\pi)) .$$

Given such a τ_0 and an arbitrary π_1-automorphism f_*, satisfying the necessary condition $r = \pm 1$, there exists a self-equivalence f, with

torsion τ_0 and induced automorphism f_*.

Because of (12ii) we have $|SK_1(\mathbb{Z}(\pi))| = h(p)$, so Theorem 1 is proven a fortiori.

3. APPENDIX ON ALGEBRAIC K-THEORY BY C. T. C. WALL

We recall and slightly modify the notation of the paper. The group $\pi = \mathbb{Z} \times \mathbb{Z}_p = \rho \times \sigma$, say, where the factors have generators t, b. Set $\sum = 1 + b + b^2 + \ldots + b^{p-1}$, and write Λ for the quotient of the group ring $\mathbb{Z}\pi$ by the ideal $\langle \sum \rangle$ generated by $\sum$. Then we assert

(i) The projection $SK_1(\mathbb{Z}\pi) \to SK_1(\Lambda)$ is an isomorphism.

(ii) The groups in (i) are isomorphic to the ideal class group of the cyclotomic field of p^{th} roots of unity.

I am indebted to Mike Stein for substantial assistance with the proof.

The proof depends on various techniques described in Bass' book [1]. We first describe these and cite some relevant results, then give the argument.

A Milnor square [11, p. 19] is a commutative diagram of ring epimorphisms

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ B & \xrightarrow{g} & B' \end{array}$$

which is bicartesian - i.e. the sequence $0 \to A \xrightarrow{\binom{f}{\phi}} A' \oplus B \xrightarrow{(\phi', -g)} B' \to 0$ (of additive groups) is exact. It is not essential for some results that f, g are surjective but it is for others. The example needed here is

$$\begin{array}{ccc} \mathbb{Z}\sigma & \longrightarrow & \mathbb{Z}\sigma/\langle \sum \rangle = \mathfrak{O}_p \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

The idea is to reduce questions about $\mathbb{Z}\sigma$ to questions about $\mathfrak{O}_p$, $\mathbb{Z}$ and $\mathbb{F}_p$.

A ring is (right) regular if it is right noetherian and finitely generated (right) modules have finite homological dimension. These rings are easier to study because questions about projective modules can be reduced to questions about the more flexible class of finitely generated modules. The rings $\mathfrak{O}_p$, $\mathbb{Z}$ and $\mathbb{Z}_p$ are all regular. If A is any regular ring, the group ring $A\rho$ (ρ infinite cyclic) is regular.

For any additive functor F on rings, and ring A, the retraction $A[t] \to A$ defined by $t \to 1$ defines a splitting $F(A[t]) = F(A) \oplus NF(A)$, say. The commutative square

$$\begin{array}{ccc} A & \longrightarrow & A[t] \\ \downarrow & & \downarrow \\ A[t^{-1}] & \longrightarrow & A[t, t^{-1}] = A\rho \end{array}$$

defines a map $F(A) \oplus N_+F(A) \oplus N_-F(A) \to F(A\rho)$ (where N_+, N_- refer to the extensions by t, t^{-1} respectively) and F is called a contracted functor if this is naturally a split injection, the other summand being denoted $LF(A)$.

The following examples of contracted functors are given by Bass [1, chapter XII].

(7.2) If F is contracted, so are NF and LF.

(7.4) K_1 is contracted, with $LK_1 = K_0$.

(7.8) For commutative rings A, $\det : K_1 \to U$ is a split epimorphism of contracted functors, so SK_1 is contracted.

It has also been shown by Quillen that for any $n \geq 1$, K_n is contracted with $LK_n = K_{n-1}$. This result is proved in [10, p. 236].

For several contracted functors F it is the case that a Milnor square (as above) gives rise to an exact sequence (see [1, XII, 8.3])

$$F(A) \to F(A') \oplus F(B) \to F(B') \to LF(A) \to LF(A') \oplus LF(B) \to LF(B').$$

This is true for $F = K_1$ (see [1, p. 481] or [11 Theorem 3.3]) and for $F = K_2$ [11, Theorem 6.4]. Moreover if the property holds for F, it also holds for NF and LF [1, XII, 8.1].

Finally, there are several vanishing theorems for regular rings. If A is regular and commutative, then $NSK_1(A) = 0$ [1, XII, 10.1]. Indeed by Quillen [12, p. 114] $NK_i(A) = 0$ for all $i \geq 1$.

We are now ready to start the proof. Applying to the Milnor square (A]) the exact sequence corresponding to the contracted functor NK_2, we obtain

$$\ldots\ NK_2(\mathbb{F}_p) \to NK_1(\mathbb{Z}\sigma) \to NK_1(\mathfrak{O}_p) \oplus NK_1(\mathbb{Z}) \to \ldots\ .$$

The outside terms vanish by regularity, hence $NK_1(\mathbb{Z}\sigma) = 0$. Now since K_1 is contracted,

$$\begin{aligned} K_1(\mathbb{Z}\pi) &= K_1(\mathbb{Z}(\rho \times \sigma)) \\ &= K_1(\mathbb{Z}\sigma[t, t^{-1}]) \\ &= K_1(\mathbb{Z}\sigma) \oplus K_0(\mathbb{Z}\sigma) \\ K_1(\Lambda) &= K_1(\mathbb{Z}\rho \otimes \mathfrak{O}_p) \\ &= K_1(\mathfrak{O}_p[t, t^{-1}]) \\ &= K_1(\mathfrak{O}_p) \oplus K_0(\mathfrak{O}_p) \end{aligned}$$

since we have just seen that the NK_1 terms vanish. Corresponding results hold for SK_1 (note that NSK_1) will vanish as a summand of NK_1); here K_0 is replaced by LSK_1, which is the subgroup $\tilde{K}_0$ corresponding to zero rank ([1, XII, 7.9]: note that $\mathbb{Z}\sigma$ and $\mathfrak{O}_p$ both have connected spectrum). Now $SK_1(\mathfrak{O}_p) = 0$ [1, p. 330] and $SK_1(\mathbb{Z}\sigma) = 0$ [1, p. 623] so we have isomorphisms

$$\begin{aligned} SK_1(\mathbb{Z}\pi) &\cong \tilde{K}_0(\mathbb{Z}\sigma), \\ SK_1(\Lambda) &\cong \tilde{K}_0(\mathfrak{O}_p)\ . \end{aligned}$$

The fact (i) that the obvious quotient map is an isomorphism now follows (see [11, p. 29] from well-known results: in the exact sequence

$$K_1(\mathbb{Z}) \oplus K_1(\mathfrak{O}_p) \to K_1(\mathbb{F}_p) \to K_0(\mathbb{Z}\sigma) \to K_0(\mathbb{Z}) \oplus K_0(\mathfrak{O}_p) \to K_0(\mathbb{F}_p),$$

$K_1(\mathfrak{O}_p)$ maps onto $K_1(\mathbb{F}_p) = \mathbb{F}_p^{\times}$, and $K_0(\mathbb{Z}) \cong K_0(\mathbb{F}_p) \cong \mathbb{Z}$. Finally the identification (ii) of $K_1(\mathfrak{O}_p)$ with the class group $\mathrm{Pic}(\mathfrak{O}_p)$ can be found on [1, p. 468] or - perhaps better - [11, Corollary 1.11].

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19 · Applications of Nielsen's reduction method to the solution of combinatorial problems in group theory: a survey

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In this paper we use the terminology and notation of [12] and [51]. One finds there the concepts and results concerning combinatorial descriptions of groups which are used (but not always further explained) below.

§1. THE NIELSEN REDUCTION METHOD IN AMALGAMATED FREE PRODUCTS AND HNN GROUPS

One of the most important methods in the theory of free groups and some similar groups, is the Nielsen reduction method. If F is the free group with free generators a, b, ... one can define a notion of free length L in F (relative to the generators a, b, ...) and a certain lexicographical ordering. The Nielsen reduction method in F concerns Nielsen transformation from systems $\{g_j\}_{j\in J}$ to systems which are shorter with respect to the length L and ordering (cf. [21], [22], [54]).

If we apply this method to a finite system $\{x_1, \ldots, x_m\}$, we arrive after a finite number of steps at a system in which no element and no inverse of an element can shorten another by more than half, and no two can shorten another to nothing - i. e. a system which possesses the Nielsen property with respect to L. Nielsen ([21], [22]) used the property for the proof that subgroups of free groups are free. To be sure, one cannot define a meaningful length function in every group (cf. [4], [8], [13]). A length satisfying certain 'natural' axioms exists essentially only in free products.

We next give a brief survey of the development of Nielsen's reduction method by Zieschang by the introduction of notions of length and order in amalgamated free products [51]. Peczynski and Reiwer carried over these

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* With thanks to Terry Wall for the English translation.

methods to HNN groups [26]. In this section

$$H = H_1 *_A H_2, \quad H_1 \neq A \neq H_2$$

denotes the non-trivial free product of the groups H_1 and H_2 with amalgam $A = H_1 \cap H_2$ and

$$K = \langle B, t \,|\, \text{rel } B, t^{-1}K_1 t \overset{\alpha}{=} K_{-1} \rangle$$

the HNN extension with basis B, stable letter t, and conjugated subgroups K_1 and K_{-1}, where $\alpha : K_1 \to K_{-1}$ is an isomorphism. We choose in each H_i $(i = 1, 2)$ a system L_i of left coset representatives of A in H_i, normalised by taking 1 to represent A. Each $x \in H$ has a unique representation $x = h_1 \ldots h_n a$ with $a \in A$, $1 \neq h_j \in L_1 \cup L_2$ and $h_{j+1} \notin L_i$ if $h_j \in L_i$. The length of x is defined as $L(x) = n$; H is (partially) ordered by length. In order to obtain results analogous to Nielsen's theorems on free groups, it is found that the ordering defined by L is too coarse. Thus - as in the free group case - we need a finer ordering of H.

For this purpose we define a symmetric normal form for elements $x \in H$. We take the inverses L_i^{-1} of the left coset representatives as a system of right coset representatives. Then each $x \in H$ has a unique representation

$$x = l_1 \ldots l_m k r_m \ldots r_1$$

with $m \geq 0$, $k \in H_1 \cup H_2$, $1 \neq l_j \in L_1 \cup L_2$, $1 \neq r_j \in L_1^{-1} \cup L_2^{-1}$, and

$l_{j+1} \notin L_i$ if $l_j \in L_i$, $r_{j+1} \notin L_i^{-1}$ if $r_j \in L_i^{-1}$

if $k \in A$, l_m and r_m belong to different H_i (if $m \geq 1$),

if $k \in H_i - A$, $l_m \notin H_i$ and $r_m \notin H_i$ (if $m \geq 1$).

We have $L(x) = \begin{cases} 2m & \text{if } k \in A \\ 2m + 1 & \text{if } k \notin A \end{cases}$. We call $l_1 \ldots l_m$ the leading half, $r_m \ldots r_1$ the rear half and k the kernel of x. One advantage of this symmetric normal form is that in forming products, cancellations can usually be reduced to free cancellation.

We have analogous normal forms in K. A representation

$$x = h_1 t^{\varepsilon_1} h_2 t^{\varepsilon_2} \dots h_n t^{\varepsilon_n} h_{n+1}, \qquad \varepsilon_i = \pm 1, \quad h_i \in B$$

in K is said to be reduced if $\varepsilon_{i+1} = -\varepsilon_i$ implies $h_{i+1} \notin K_{\varepsilon_{i+1}}$. Now choose normalised systems R_1, respectively R_{-1}, of left coset representatives of K_1, respectively K_{-1} in B. Then each element $x \in K$ may be uniquely represented as

$$x = l_1 t^{\varepsilon_1} l_2 t^{\varepsilon_2} \dots l_n t^{\varepsilon_n} b$$

with $\varepsilon_i = \pm 1$, $b \in B$, $l_i \in R_{\varepsilon_i}$ and $\varepsilon_i = \varepsilon_{i+1}$ when $l_{i+1} = 1$. The length $L(x)$ is defined to be n. Again we take the inverses R_1^{-1}, R_{-1}^{-1} as systems of right coset representatives. Each $x \in K$ has a reduced representation

$$x = l_1 t^{\varepsilon_1} \dots l_m t^{\varepsilon_m} k t^{\eta_m} r_m \dots t^{\eta_1} r_1$$

with $m \geq 0$, $l_i \in R_{\varepsilon_i}$, $r_i \in R_{-\eta_i}^{-1}$, and $k = h_1 t^{\varepsilon} h_2$ $(h_1, h_2 \in B)$ if $L(x)$ is odd, $k \in B$ if $L(x)$ is even. Then $l_1 t^{\varepsilon_1} \dots l_m t^{\varepsilon_m}$ is called the leading half, $t^{\eta_m} r_m \dots t^{\eta_1} r_1$ the rear half and k the kernel of x.

We now introduce orderings on H and K, beginning with H. For our applications, the groups will be countable. This is no restriction if one considers finitely generated subgroups of H and K (in particular, Theorems 1.1 to 1.5 hold for arbitrary H and K). Enumerate the system L_1 (similarly L_2) as $\{x_i : i \in \mathbb{N}\}$, and order it correspondingly. Let the elements of L_1 precede those of L_2. Then we order (for each m) the products $l_1 \dots l_m$ of coset representatives (where $1 \neq l_j \in L_1 \cup L_2$ and $l_{j+1} \notin L_i$ when $l_j \in L_i$) lexicographically. The following properties of this ordering are used in the proof:

(1) If $l_1 \dots l_m < l'_1 \dots l'_m$ then for any permitted l_{m+1}, l'_{m+1} we have $l_1 \dots l_{m+1} < l'_1 \dots l'_{m+1}$.

(2) Each product $l_1 \dots l_m$ has only finitely many predecessors of the form $l_1 \dots l_{m-1} l'_m$ (where $l'_m \in L_i$ if $l_m \in L_i$).

We define an ordering on the products of coset representatives in the L_i^{-1} by taking inverses.

We now proceed similarly in K. First choose total orders of the systems R_1, R_{-1} of coset representatives, and then order the products $l_1 t^{\varepsilon_1} \ldots l_m t^{\varepsilon_m}$ using the lexicographical ordering of the sequence $(l_1, \ldots, l_m)$.

Now we extend this ordering to the set of pairs $\{g, g^{-1}\}$, $g \in H$, respectively, K, where the notation is so chosen that the leading half of g precedes that of g^{-1} with respect to the ordering $<$. Then we set $\{g, g^{-1}\} \prec \{g', g'^{-1}\}$ if either $L(g) < L(g')$ or $L(g) = L(g')$ and the leading half of g strictly precedes that of g', or $L(g) = L(g')$, the leading halves of g and g' coincide, and the leading half of g^{-1} precedes that of g'^{-1}. Thus if $\{g, g^{-1}\} \prec \{g', g'^{-1}\}$ and $\{g', g'^{-1}\} \prec \{g, g^{-1}\}$, g and g' differ only in the kernel. A system $\{g_j\}_{j \in J}$ in H or K is called shorter than a system $\{g'_j\}_{j \in J}$ if $\{g_j, g_j^{-1}\} \prec \{g'_j, g'^{-1}_j\}$ holds for all $j \in J$, but for at least one j, $\{g'_j, g'^{-1}_j\} \prec \{g_j, g_j^{-1}\}$ fails. A system $\{g_j\}_{j \in J}$ is said to be minimal with respect to $\prec$ if there is no system freely equivalent to $\{g_j\}_{j \in J}$ which is shorter.

The Nielsen reduction method in H or K now refers to Nielsen transformation from systems $\{g_j\}_{j \in J}$ to shorter systems, and the investigation of minimal systems; and thus copies the Nielsen reduction method in free groups. An analysis of the results of Zieschang [51] for H produces

Theorem 1.1 (Theorem 1 of [51] and Corollary 2 of [34]). A finite system $\{x_1, \ldots, x_n\} \subset H$ is Nielsen equivalent to a system $\{y_1, \ldots, y_n\}$ satisfying one (at least) of the following:

(i) Each $w \in \langle y_1, \ldots, y_n \rangle$ has a representation $w = y_{u_1}^{\varepsilon_1} \ldots y_{u_q}^{\varepsilon_q}$, $\varepsilon_i = \pm 1$, where $\varepsilon_i = \varepsilon_{i+1}$ if $u_i = u_{i+1}$ with $L(y_{u_i}) \leq L(w)$ for $i = 1, \ldots, q$.

(ii) There is a product $a = y_{u_1}^{\varepsilon_1} \ldots y_{u_q}^{\varepsilon_q}$, $a \neq 1$, with $y_{u_i} \in A$ $(i = 1, \ldots, q)$ and in one factor H_j an element $x \notin A$ with $xax^{-1} \in A$.

(iii) For some $p \geq 1$, p of the y_i lie in a subgroup of H conjugate to H_1 or H_2, not all in A, and some product in them is conjugate to an element of A different from 1.

(iv) For some $g \in H$, $y_1 \notin gAg^{-1}$ but there exists $m \in \mathbb{N}$ with $1 \neq y_1^m \in gAg^{-1}$.

The Nielsen transformation can be chosen in finitely many steps so that $\{y_1, \ldots, y_n\}$ is shorter than $\{x_1, \ldots, x_n\}$ or the lengths of the elements remain the same.

Remarks. (1) If $\{x_1, \ldots, x_n\}$ is a system of generators for H, it follows in case (i) that $L(y_i) \leq 1$ for $i = 1, \ldots, n$. In case (iii) we find that $p \geq 2$, since in the case of systems of generators conjugation defines a Nielsen transformation.

(2) If we merely ask for a combinatorial description of $\langle x_1, \ldots, x_n \rangle$ by generators and relations, we find - after appropriate conjugation - that again $p \geq 2$ in case (iii).

As immediate consequences of Theorem 1.1 we have

Corollary 1 [11]. Suppose A malnormal in H_1 and in H_2 and N a subgroup of rank 2 in G. Then N is either the free product of two cyclic groups or conjugate to a subgroup of H_1 or of H_2.

Corollary 2. If H has rank 2, a generating system $\{x_1, x_2\}$ for H is Nielsen equivalent to a system $\{y_1, y_2\}$ for which case (i), (ii) or (iv) occurs.

A closer investigation of groups H of rank 2 yields further

Theorem 1.2 ([42], [38]). Suppose H of rank 2 and A cyclic. Further suppose $\langle x_1 \rangle$ malnormal in H for every system $\{x_1, x_2\}$ of generators of H with $\langle x_1 \rangle \cap A \neq \{1\}$. Then every generating system $\{x_1, x_2\}$ of H is Nielsen equivalent to a system $\{y_1, y_2\}$ with $y_1 \in H_1$ and $y_2 \in H_2$. In particular H has a combinatorial definition

$$H = \langle a, b \mid R_1(a^m, b) = R_2(a^m, b) = \ldots = 1 \rangle,$$

where $\langle a \rangle$ is malnormal in H, $m \geq 2$, and if ord a is finite, m divides ord a. Further, $\{x_1, x_2\}$ is Nielsen equivalent to a system $\{a^\gamma, z\}$ with $z \in \langle a^m, b \rangle$ and $\gamma = 1$ if ord $a = \infty$, $1 \leq \gamma <$ ord a, $(\gamma, \text{ord } a) = 1$ if ord $a < \infty$.

Remark. A further direct consequence of Theorem 1.1 is Grusko's theorem for finitely generated free products (the case $A = \{1\}$). However, the proof of Theorem 1.1 gives no definite hint that Grusko's theorem breaks down for finitely generated amalgamated free products $H_1 *_A H_2$. In the proof one studies minimal systems $\{x_1, \ldots, x_n\}$ such that one can find $w \in \langle x_1, \ldots, x_n \rangle$ and a representation $w = x_{u_1}^{\varepsilon_1} \ldots x_{u_q}^{\varepsilon_q}$ $(\varepsilon_i = \pm 1,\ \varepsilon_i = \varepsilon_{i+1}$ if $u_i = u_{i+1})$ with $L(x_{u_j}) > L(w)$ for some j; one seeks possible reasons for this, and then lists the reasons. One cannot deduce from the proof to what extent (ii) to (iv) are genuine restrictions. Theorem 1.2 shows precisely that for the generating pairs $\{x_1, x_2\}$ which appear there, case (iv) occurs; nevertheless we find that Grusko's theorem holds for such pairs. However examples are considered in [34] which show that none of the cases (ii) to (iv) can be omitted. The following example already shows that Grusko's theorem does not hold in general for amalgamated free products.

Let $H_1 = \langle s_1, s_2 | s_1^2 = s_2^2 = 1 \rangle$, $H_2 = \langle s_3, s_4 | s_3^2 = s_4^m = 1 \rangle$ with $m > 2$ odd, $A = \langle s_1 s_2 = (s_3 s_4)^{-1} \rangle$ and $H = H_1 *_A H_2$. Then $H = \langle s_1 s_2, s_1 s_3 \rangle$ and $\{s_1 s_2, s_1 s_3\}$ is not Nielsen equivalent to a system $\{x_1, x_2\}$ with $x_i \in H_1 \cup H_2$ (cf. [10], [25], [34]). This also gives a counterexample to the plausible conjecture

$$\text{Rk}(H_1 *_A H_2) \geq \text{Rk}H_1 + \text{Rk}H_2 - \text{Rk}A.$$

For K, Peczynski and Reiwer obtained the analogous result by considerations corresponding to those in [51].

Theorem 1.3 [26]. Each finite system $\{x_1, \ldots, x_n\} \subset K$ is Nielsen equivalent to a system $\{y_1, \ldots, y_n\}$ for which one of the following holds:

(i) Each $w \in \langle y_1, \ldots, y_n \rangle$ has a representation $w = y_{u_1}^{\varepsilon_1} \ldots y_{u_q}^{\varepsilon_q}$

$(\varepsilon_i = \pm 1,\ \varepsilon_i = \varepsilon_{i+1}$ if $u_i = u_{i+1})$ with $L(y_{u_i}) \leq L(w)$ for $1 \leq i \leq q$.

(ii) Some subgroup of K conjugate to B contains p of the y_i, and some product in them is conjugate to a non-trivial element of K_1.

As first application of Theorem 1.3 one has a new proof of a reduction theorem of Pride:

Theorem 1.4 ([26], [27]). Suppose K of rank 2 and both K_1 and K_{-1} malnormal in B. Then any system $\{x_1, x_2\}$ of generators of K is Nielsen equivalent to a system $\{t^{\varepsilon}g, h\}$ with $\varepsilon = \pm 1$, $g \in B$ and $h \in K_1 \cup K_{-1}$.

As further direct application of Theorem 1.1 and Theorem 1.3 one has a new proof of a subgroup theorem of H. Neumann:

Theorem 1.5 ([19]). Let U be a finitely generated subgroup of H with $gUg^{-1} \cap A = \{1\}$ for all $g \in H$, or of K with $gUg^{-1} \cap K_\eta = \{1\}$ $(\eta = \pm 1)$ for all $g \in K$. Then $U = F * (*_{i \in I} G_i)$, where F is a free group and each G_i is conjugate to a subgroup of H_1 or of H_2 respectively to a subgroup of B.

A proof for K is given in [26]. For H the hypotheses say that none of the cases (ii)-(iv) of Theorem 1.1 can occur. Hence case (i) of Theorem 1.1 must occur. But this case states directly that each relation in U is a consequence of relations from H_1 or from H_2.

A close investigation of cases (ii) to (iv) of Theorem 1.1 respectively (ii) of Theorem 1.3 leads to further applications of these Theorems. We proceed to such applications in §2 and §3.

§2. THE ISOMORPHISM PROBLEM FOR ONE-RELATOR GROUPS

Let $G = \langle a_1, \ldots, a_n | R = 1\rangle$ be a group with one defining relation. We say that the isomorphism problem for G is solvable if there is an algorithm allowing one to decide in finitely many steps whether a given further group $H = \langle a_1, \ldots, a_n | S = 1\rangle$ with one defining relation is isomorphic to G or not. As application of Theorem 1.4, Pride obtained

Theorem 2.1 ([27], [28]). Let $G = \langle t, a \,|\, P^\delta = 1\rangle$, $\delta \geq 2$. Then the isomorphism problem for G is solvable.

Pride showed in [28] that one may assume without loss of generality that P is cyclically reduced, is not a proper power in the free group on t and a, and the sum of the exponents of t in P is zero (cf. [12]). If P is 1 or a primitive element, the assertion of Theorem 2.1 is clear. Now suppose P neither trivial nor primitive. Set $a_i = t^{-i}at^i$ $(i \in \mathbb{Z})$. Let Q be the word expressing P in terms of the a_i. Let m, respectively M, be the least, respectively greatest, value of i for which a_i appears in Q. Then G is an HNN group

$$G = \langle a_m, \ldots, a_M, t \,|\, Q^\delta = 1, t^{-1}a_i t = a_{i+1} \ (i = m, \ldots, M-1)\rangle.$$

The conjugate subgroups $K_1 = \langle a_m, \ldots, a_{M-1}\rangle$ and $K_{-1} = \langle a_{m+1}, \ldots, a_M\rangle$ are malnormal in the basis $B = \langle a_m, \ldots, a_M \,|\, Q^\delta = 1\rangle$ (cf. [12], [18], [20]). Now apply Theorem 1.4 to see that a generating system $\{x_1, x_2\}$ for G is Nielsen equivalent to a system $\{tg, h\}$, $g \in B$, $h \in K_{-1}$. This pair $\{tg, h\}$ is Nielsen equivalent to $\{t, a\}$ (cf. [28]). Now Theorem 2.1 follows by using the Whitehead algorithm (cf. [14], [55], [56]). As consequence of Theorem 2.1 we have

Corollary. Suppose G as in Theorem 2.1. Then (a) G is hopfian (cf. [27, Theorem 2]). (b) The automorphism group of G is finitely generated (cf. [9], [16]).

Remark. Suppose G as in Theorem 2.1 and not decomposable as a free product of cyclic groups. Then each automorphism of G can be lifted to an automorphism of the free group of rank 2. Thus the determination of the automorphisms of G is equivalent to the determination of the stabilisers and symmetries of the element $P(t, a)^\delta$ in the free group on t and a. Such problems are thoroughly investigated in [47]. This and further work leads to various results on the automorphism group Aut G of G, for example the following. If G/G' is free abelian of rank 2 i. e. if $P(t, a) \in G'$, then Aut G is complete, i. e. it has trivial centre and every automorphism of Aut G is inner.

Theorem 1.1 can also be applied to the solution of the isomorphism problem for certain one-relator groups, particularly for those that can be decomposed as a nontrivial free product with cyclic amalgam. Typical examples of such groups are

$$G = \langle a_1, \ldots, a_p, b_1, \ldots, b_q \mid W(a_1, \ldots, a_p)V(b_1, \ldots, b_q)^\gamma = 1\rangle$$
$$\gamma \geq 1$$

with V and W non-trivial. Let G denote such a group for the remainder of this paragraph.

Theorem 2.2 ([37]). If $\gamma \geq 2$, the isomorphism problem for G is solvable.

If G is decomposable as a free product of cyclic groups, the assertion of Theorem 2.2 is clear. Now suppose G not decomposable as a free product. We display G as amalgamated free product $H_1 *_A H_2$ with

$$H_1 = \langle a_1, \ldots, a_p\rangle, \quad H_2 = \langle s, b_1, \ldots, b_q \mid s^\gamma = 1\rangle \quad \gamma \geq 2$$

and A is generated by $W^{-1} = Vs$. Now let $\{x_1, \ldots, x_{p+q}\}$ be a minimal system of generators of G. By Theorem 1.1 we can consider without loss of generality case (iii) of 1.1. With the help of [2] and [44] one finds: this case (iii) occurs for H_1, and we take $x_1, \ldots, x_p \in H_1$. By refactorising it follows similarly that we can also suppose $x_{p+1}, \ldots, x_{p+q} \in H_2$. But then $\{x_1, \ldots, x_{p+q}\}$ is Nielsen equivalent to $\{a_1, \ldots, a_p, b_1, \ldots, b_q\}$ and the result again follows using the Whitehead algorithm (cf. [14], [55], [56]).

As in [28] follows

Corollary. A group G as in Theorem 2.2 is hopfian, and Aut G is finitely generated.

Now suppose $\gamma = 1$. This case, i.e. the torsion free case, is more difficult. Here it may happen that G possesses infinitely many Nielsen equivalence classes. Even for $\gamma = 1$ the isomorphism problem for G is solvable if

$$W = (a_1^{\alpha_1} \dots a_n^{\alpha_n}[a_{n+1}, a_{n+2}] \dots [a_{p-1}, a_p])^{\alpha}, \quad 0 \le n \le p,\ \alpha_i \ge 2,\ \alpha \ge 1$$

$$V = (b_1^{\beta_1} \dots b_m^{\beta_m}[b_{m+1}, b_{m+2}] \dots [b_{q-1}, b_q])^{\beta}, \quad 0 \le m \le q,\ \beta_i \ge 2,\ \beta \ge 1$$

(where $\alpha = 1$ if $p = 1$ and $\beta = 1$ if $q = 1$) are alternating products. These groups are of interest because they include the fundamental groups of closed surfaces and groups of torus knots as special cases. We suppose for the rest of this section that W and V are such alternating products. The following result, which is important in the discussion of case (iii) of Theorem 1.1, is the essential tool for the solution of the isomorphism problem. It is proved using the Nielsen reduction method in free groups.

Theorem 2.3 [39]. Let F be the free group on $a_1, \dots, a_p$ ($p \ge 1$). Let $\{x_1, \dots, x_r\}$ ($r \ge 1$) be any system in F, and $W^u \in \langle x_1, \dots, x_r \rangle$ for some $u \ne 0$ with W an alternating product as above. Then one of the following cases occurs:

(a) $\{x_1, \dots, x_r\}$ is Nielsen equivalent to a system $\{y_1, \dots, y_r\}$ with $y_1 = zW^{\rho}z^{-1}$, $\rho > 0$, $z \in F$.

(b) We have $r \ge p$, and $\{x_1, \dots, x_r\}$ is Nielsen equivalent to a system $\{y_1, \dots, y_r\}$ with $y_1 = za_i^{\gamma_i}z^{-1}$, $\gamma_i \mid \alpha_i$ ($i = 1, \dots, n$), $y_j = za_jz^{-1}$ ($j = n+1, \dots, p$), $z \in F$.

Corollary [39]. Suppose $(\alpha_1, \dots, \alpha_n) \ge 2$. If ϕ is an endomorphism of F which leaves W fixed, then the ϕ is an automorphism of F.

Automorphisms of F which fix a particular word W were studied in [50]. The above Theorem 2.3 extends some results of [48] (the question of when two alternating products are related is considered there). Using Theorem 2.3 one can show:

Theorem 2.4 (cf. [23], [34], [35], [39], [51]). If $p \ge 2$ or $q \ge 2$ then there are only finitely many distinct Nielsen equivalence classes of minimal generating systems $\{x_1, \dots, x_{p+q}\}$, and for each system there is a presentation of G with one defining relation.

The solution of the isomorphism problem for G follows from this (if $p \geq 2$ or $q \geq 2$).

It follows as a corollary that if $p \geq 2$ or $q \geq 2$ every automorphism of G is induced by an automorphism of the free group of rank $p + q$. This gives a new proof of the result of Nielsen, that every automorphism of the fundamental group of a closed orientable (respectively nonorientable) surface of genus g is induced by an automorphism of the free group of rank $2g$ (respectively g) (cf. also [23], [34], [51], [54]). Moreover the automorphism group of G is finitely generated.

If $p = q = 1$ and $\alpha_1 + \beta_1 = 4$, there is only one Nielsen equivalence class of generating systems $\{x_1, x_2\}$, and there is nothing to prove.

If $p = q = 1$ and $\alpha_1 + \beta_1 \geq 5$, there are infinitely many distinct Nielsen equivalence classes of generating systems $\{x_1, x_2\}$ (cf. [35], [52]), and each generating system $\{x_1, x_2\}$ is freely equivalent to just one system $\{a_1^{\gamma_1}, b_1^{\gamma_2}\}$ with $(\gamma_1, \gamma_2) = (\gamma_1, \alpha_1) = (\gamma_2, \beta_1) = 1$ and $1 \leq \gamma_1 \leq \frac{1}{2}\gamma_2\alpha_1$, $1 \leq \gamma_2 \leq \frac{1}{2}\gamma_1\alpha_2$ (cf. [52]). By [6], such a system $\{a_1^{\gamma_1}, b_1^{\gamma_2}\}$ can only give rise to a one-relator presentation of G if $\gamma_1 = 1$ or $\gamma_2 = 1$. The solution of the isomorphism problem for G in the case $p = q = 1$ now follows. Here again, every automorphism of G is induced by an automorphism of the free group of rank 2 [43].

Remarks. (1) Theorem 2.1 and Theorem 2.2 lead one to make the following conjecture: The isomorphism problem is solvable for one-relator groups with torsion.

This is not the case for torsion-free one-relator groups. Recently Brunner [3] found an example of a torsion free group with infinitely many Nielsen-inequivalent one-relator presentations. This example also contradicts a conjecture of Magnus about the possible one-relator presentations of a group (cf. [14, p. 401]). The above groups with $\gamma = 1$ yield further counterexamples to this conjecture (cf. [17], [34], [35], [51], [52]).

(2) In [5], Theorem 1.1 is applied to describe generators and presentations of a one-relator group with centre.

(3) The methods developed for the proof of Theorems 2.1 and 2.2 yield some results about subgroups of one-relator groups with torsion (cf. [29], [36]). In particular,

Suppose $H = \langle a_1, \ldots, a_n | R^\delta = 1\rangle$ with R cyclically reduced and $\delta \geq 2$, and let U be a subgroup of H of rank 2. Then U is either a free product of two cyclic groups or a one-relator group with torsion ([29]).

This result is not generally valid for subgroups U of rank 3: indeed neither for H as in Theorem 2.1 nor for H as in 2.2. However we do have the following result for some groups G as in Theorem 2.2.

Theorem 2.5 ([42]). Let $G = \langle a_1, \ldots, a_p, b_1, \ldots, b_q | (\overline{W}(a_1, \ldots, a_p)\overline{V}(b_1, \ldots, b_q))^\gamma = 1\rangle$, $\gamma \geq 1$, $p \geq 2$, $q \geq 2$, $\overline{W}$ and $\overline{V}$ non-trivial. Suppose $\overline{W}$, respectively $\overline{V}$, not a proper power in $\langle a_1, \ldots, a_p\rangle$ respectively $\langle b_1, \ldots, b_q\rangle$. Then every subgroup $U \subset G$ of rank three is a free product of cyclic groups.

This theorem extends a result of B. Baumslag [1] about subgroups $U \subset G$ of rank two. The corresponding assertion for subgroups of rank four is not true (cf. [36] and [42]).

§3. AUTOMORPHISMS OF DISCONTINUOUS PLANE GROUPS

In this paragraph we investigate the groups

$$G = \langle s_1, \ldots, s_m, a_1, \ldots, a_p | s_1^{\gamma_1} = \ldots = s_m^{\gamma_m} = s_1 \ldots s_m(a_1^{\alpha_1} \ldots a_n^{\alpha_n}[a_{n+1}, a_{n+2}] \ldots [a_{p-1}, a_p])^\gamma = 1\rangle,$$

with $\gamma_1 \geq 2$, $\alpha_j \geq 2$, $\gamma \geq 1$, $0 \leq n \leq p$, $m \geq 3$ if $p = 0$ and $m \geq 2$ if $p = 1$. These groups are of interest because they include as special cases the discontinuous plane groups (without reflections) with compact fundamental region. In [11], [38] and [53] the question is considered (among others), which of the groups G are decomposable as amalgamated free product. Theorems 1.1 and 2.3 together with [25, Theorem 1] yield the following result about the rank problem for G.

Theorem 3.1 (cf. [25], [34], [39]). *The rank* G *is*

(a) p *if* $m = 0$, $p > 1$,

(b) $m - 2$ *if* $p = 0$, m *is even, and all* γ_i *equal* 2 *except for one, which is odd,*

(c) $p + m - 1$ *in all other cases.*

Case (b) of Theorem 3.1 is unexpected; here the rank differs from what one would anticipate on geometrical grounds. If G is a discontinuous plane group, without reflections, and having a compact fundamental region, then for each connected closed fundamental region F of G there are at least p (for $m = 0$) respectively $p + m - 1$ (for $m \geq 1$) pairs x, $x^{-1} \in G$ for which $xF \cap F$ has dimension ≥ 1 (cf. [25]). Thus Theorem 3.1 asserts that the rank of G need not coincide with the geometrical rank.

By using the Reidemeister-Schreier algorithm one can obtain from Theorem 3.1 a lower bound for the rank of a discontinuous plane group which has a compact fundamental region and contains reflections (cf. [46]).

Now suppose if $p = 0$ that

$$m - 2 - \sum_{i=1}^{m} \gamma_i^{-1} > 0$$

(in particular G is infinite). If $m = 3$ and $p = 0$ it follows from [40] that every automorphism of G is induced by an automorphism of the free group of rank 2 (cf. also [54]). If $m \leq 1$ and $p \geq 2$ G is a one-relator group, and it follows from §2 that every automorphism of G is induced by an automorphism of the free group of rank p. In all other cases the corresponding result follows by applying the Nielsen reduction method in amalgamated free products together with Theorem 2.3 and [25, Theorem 1] To summarise:

Theorem 3.2. *If* G *has rank* r, *every automorphism of* G *is induced by an automorphism of the free group of rank* r.

This theorem extends and completes a result of Zieschang [49], who showed that if G is a discontinuous plane group, every automorphism of G is induced by an automorphism of the free group of rank $p + m$.

The case (b) of Theorem 3.1, where rank $G = m - 2$, is particularly interesting. Here there is just one Nielsen equivalence class of generating systems $\{x_1, \ldots, x_{m-2}\}$ (cf. [10], [34]). If in particular $m = 4$, so rank $G = 2$, it is also true that every automorphism of the free group of rank 2 induces an automorphism of G [10]. Thus one can naturally identify Aut G with an epimorphic image of the automorphism group of the free group of rank 2; i. e. G has a quasifree presentation (cf. [12]).

Further results on the characterisation of minimal generating systems of G are given in [10], [24], [34] and [39].

§4. A NIELSEN REDUCTION METHOD IN LINEAR GROUPS OVER $\mathbb{R}$

By a theorem of Tits [45], every finitely generated linear group over a commutative field either has a solvable subgroup of finite index or contains a free subgroup of rank two. For subgroups of $SL(2, \mathbb{R})$ this theorem can be extended by use of a Nielsen reduction process in linear groups. In describing this method we restrict ourselves to the case of two-generator subgroups of $SL(2, \mathbb{R})$. Consider then a subgroup $G = \langle A, B \rangle$ of $SL(2, \mathbb{R})$ with $\mathrm{Tr}A = x$, $\mathrm{Tr}B = y$ and $\mathrm{Tr}AB = z$.

It is easy to see that in general

$$\mathrm{Tr}[A, B] = x^2 + y^2 + z^2 - xyz - 2.$$

If $\mathrm{Tr}[A, B] = 2$, G is reducible i. e. A and B (considered as linear fractional transformations) have at least one common fixed point. If two of x, y and z vanish then G is isomorphic modulo $\pm I$ to the infinite dihedral group. In both cases, G contains a solvable subgroup of finite index.

From now on we suppose $\mathrm{Tr}[A, B] \neq 2$ and that at most one of x, y and z vanishes. Then G does not have a solvable subgroup of finite index. Let E_G be the set of all pairs $\{U, V\}$ freely equivalent to $\{A, B\}$. Then $G = \langle U, V \rangle$ and $\mathrm{Tr}[U, V] = \mathrm{Tr}[A, B]$ so the ternary form $f(x, y, z) = x^2 + y^2 + z^2 - xyz$ is invariant by automorphisms of the free group of rank two.

Let $L_G = \{(\mathrm{Tr}U, \mathrm{Tr}V, \mathrm{Tr}UV) \mid \{U, V\} \in E_G\}$ and $M_G = \{\mathrm{Tr}U \mid \{U, V\} \in E_G$ for some $V \in G\}$. Then M_G is a discrete

subset of $\mathbb{R}$, and starting from the given triple $(x, y, z) \in L_G$ we can obtain all triples $(u, v, w) \in L_G$ by repeated application of the following birational transformations:

$$O_1 : u \to v,\ v \to u,\ w \to w$$
$$O_2 : u \to w,\ v \to u,\ w \to v$$
$$O_3 : u \to u,\ v \to v,\ w \to uv - w.$$

Indeed, each automorphism of the free group of rank 2 induces in a natural way such a birational transformation, since $TrRTrS\text{-}TrRS=TrRS^{-1}$ for $R, S \in SL(2, \mathbb{R})$. Further, the permutation group H generated by O_1, O_2 and O_3 is isomorphic to $PGL(2, \mathbb{Z})$ and operates discontinuously on L_G (cf. [7], [33], [41]). The proof is essentially based on the following simple observation:

If $R, S \in SL(2, \mathbb{R})$ with $|Tr\, R| \leq 2$, then $Tr[R, S] \geq 2$. Suppose now, $x, y \geq 0$ and write E_G^+ for the set of pairs $\{U, V\} \in E_G$ with $0 \leq TrU \leq TrV \leq TrUV$. Then E_G^+ is non-empty, for if $z < 0$ we replace z by $xy - z > 0$ (replace A by A^{-1}); then apply a permutation (product in O_1 and O_2) to suppose $0 \leq x \leq y \leq z$.

We now introduce an order $\prec$ on E_G^+. If $\{U, V\}$, $\{R, S\} \in E_G^+$, set $\{U, V\} \prec \{R, S\}$ if $TrU + TrV + TrUV < TrR + TrS + TrRS$. Now consider Nielsen transformations from pairs $\{R, S\} \in E_G^+$ to pairs $\{U, V\} \in E_G^+$ which are shorter with respect to the ordering $\prec$. Since M_G is a discrete subset of $\mathbb{R}$, we arrive after finitely many steps at a pair $\{U, V\} \in E_G^+$ minimal with respect to $\prec$, i.e. a pair $\{U, V\} \in E_G^+$ with $TrUV^{-1} < 0$ (if $Tr[A, B] > 2$) respectively $TrUV \leq \frac{1}{2} TrUTrV$ (if $Tr[A, B] < 2$). This Nielsen reduction process now leads in conjunction with a theorem of Majeed [15] to

Theorem 4.1 [41]. *Let $G = \langle A, B\rangle \subset SL(2, \mathbb{R})$ with $TrA = x$, $TrB = y$, $TrAB = z$ and $x^2 + y^2 + z^2 - xyz \neq 4$. Suppose at most one of x, y, z is zero. Then G has a generating system $\{U, V\}$ such that $\langle U^n, V^n\rangle$ is a discrete free group of rank 2 for n sufficiently large.*

Remark. By applying this Nielsen reduction method we can obtain further results for groups $G = \langle A, B\rangle \subset SL(2, \mathbb{R})$. We have simple

necessary and sufficient conditions for G to be discrete. We can further classify discrete groups $G = \langle A, B \rangle$ up to conjugacy in $GL(2, \mathbb{R})$ (cf. [10], [30], [31], [32], [33], [40]). Using the ternary form $f(x, y, z) = x^2 + y^2 + z^2 - xyz$ we obtain a description of the Teichmüller space for certain discrete groups $G = \langle A, B \rangle \subset SL(2, \mathbb{R})$ as a real, affine algebraic set defined over $\mathbb{Z}$ on which the group $H \cong PGL(2, \mathbb{Z})$ generated by O_1, O_2 and O_3 acts discontinuously. In particular, H acts discontinuously on the set $\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x, y, z \text{ and } x^2+y^2+z^2 - xyz=b\}$ for $b \leq 0$ in $\mathbb{R}$; and a fundamental region for H is given by the subset where $2 < x \leq y \leq z \leq \frac{1}{2}xy$ (cf. [7], [30], [32], [33]). A general theorem about the description of Teichmüller space by a real affine algebraic set defined over $\mathbb{Z}$ is given in [7].

This Nielsen reduction process also leads to

Theorem 4.2 (cf. [34]). A non-elementary subgroup of $SL(2, \mathbb{R})$ is discrete if and only if each of its cyclic subgroups is discrete.

Here a subgroup G of $SL(2, R)$ is said to be elementary if the commutator of any two elements of infinite order has trace 2.

A proof of this theorem has also been obtained by T. Jørgensen.

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20 · Chevalley groups over polynomial rings

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Let $\underline{G}$ be a Chevalley group (scheme) defined over $\mathbb{Z}$, simple and simply-connected, and $A = k[t]$ the ring of polynomials over a field k. We shall describe an action of the group $\Gamma = \underline{G}(k[t])$ on an appropriate contractible space, and deduce from that information about the presentations and the homology of the group Γ.

1. REDUCTION THEORY ON BUILDINGS

Let $\underline{G}$ and A be as above, and call

$K = k(t)$ the fraction field of A, G the group $\underline{G}(K)$,

ω the valuation defined on K by $\omega(u/v) = \deg v - \deg u$, $\mathcal{O}$ the ring of integers for this valuation ($\mathcal{O} \neq A$),

$\underline{T}$ a maximal torus in $\underline{G}$, ϕ the set of roots of $\underline{G}$ with respect to $\underline{T}$, and $S \subset \phi$ a set of simple roots,

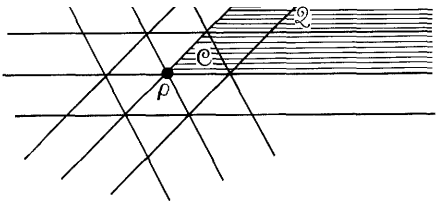
$\mathcal{T}$ the (affine) Bruhat-Tits building associated to G and ω [1],

$\mathcal{A}$ the standard apartment associated to T, ϕ the vertex fixed by $\underline{G}(\mathcal{O})$, $\mathcal{Q}$ the 'quartier' with vertex ϕ associated to S, $\mathcal{C}$ the fundamental chamber containing ϕ,

$\underline{G} \subset SL_n$ an imbedding of $\underline{G}$ in a special linear group such that $\underline{T}$ is diagonal and $\Gamma = SL_n(A) \cap G$,

$j : \mathcal{T} \to \mathcal{T}'$ an injection of $\mathcal{T}$ into the building $\mathcal{T}'$ of $SL_n(K)$, compatible with the preceding imbedding, mapping $\mathcal{A}$ into the standard apartment of $\mathcal{T}'$ and multiplying the distances by a fixed constant (cf. [1], 9-1-19, c)).

The case of SL_3 is drawn below:



Theorem 1. The set $\mathcal{Q}$ is a simplicial fundamental domain for the action of Γ on $\mathcal{T}$. In other words, any simplex of $\mathcal{T}$ is equivalent by Γ to a unique simplex of $\mathcal{Q}$.

Proof. This result, as well as theorems 2, 3 and 4 below, is a generalisation of theorems of J. P. Serre [11] in the case of SL_2 (I'd like also to thank J. Tits for his help in this proof).

1.1 Description of the isotropy group Γ_x of a vertex x of $\mathcal{Q}$ in Γ

Since $\Theta \cap A = k$, one has $\Gamma_\phi = \underline{G}(k)$.

When $x \in \mathcal{Q} - \{\phi\}$, and if $[x[$ is the half-line of origin x and direction $\vec{\phi x}$, one has $\Gamma_x = \Gamma_{[x[}$. It is enough to prove this for SL_n, since j respects the geodesics. In that case the stabilizer of x in G is, with the notations of [1] 10-2-8, p. 238,

$$P_x = \{g = (g_{ij}) / \omega(g_{ij}) + a_j(x) - a_i(x) \geq 0\},$$

and:

$$P_{[x[} = \left\{ g = (g_{ij}) \;\middle/\; \begin{array}{l} g_{ij} = 0 \text{ if } a_j(x) - a_i(x) < 0 \\ \omega(g_{ij}) + a_j(x) - a_i(x) \geq 0 \text{ if } a_j(x) - a_i(x) \geq 0 \end{array} \right\}$$

But $\omega(A - \{0\}) \leq 0$, and this implies the result.

In general, let x_a denote the one parameter subgroup associated to the root $a \in \Phi$. The stabilizer of $[x[$ in G is the semi-direct product of

$$Z_x(\Theta) = \underline{T}(\Theta) \cdot \langle x_a(\Theta),\ a(x) = 0,\ a \in \Phi \rangle$$

with

$$U_x(K) = \langle x_a(u),\ u \in K,\ \omega(u) \geq a(x) > 0,\ a \in \Phi \rangle$$

(see [1], 6.1.3.b) and 7.1). Therefore $\Gamma_x = Z_x(k).U_x(A)$, where

$$Z_x(k) = \underline{T}(k).\langle x_a(k),\ a(x) = 0 \rangle$$

and $U_x(A) = \langle x_a(u),\ u \in k[t],\ d^\circ(u) \leq a(x),\ a(x) > 0 \rangle$ (cf. [14], p. 114).

1.2 Action of Γ_x on the link of x in $\mathcal{T}$

If L_x is this link, we prove that $\Gamma_x.(L_x \cap \mathcal{Q}) = L_x$. Actually, from [1], 7-2-7, L_x is the spherical Tits building of a quotient $\overline{G}_x$ of P_x. Furthermore, $\mathcal{C} \cap L_x$ is an apartment of this building and Γ_x projects into $\overline{G}_x$ with image the parabolic subgroup associated to the roots which are zero on x; this group admits precisely $L_x \cap \mathcal{Q}$ as fundamental domain.

1.3 Two distinct points in $\mathcal{Q}$ are not equivalent by Γ

If two points of $\mathcal{Q}$ are equivalent under Γ, two chambers containing them are equivalent under Γ (because of 1.2)).

Two chambers in $\mathcal{C}$ are then equivalent under Γ and by a translation τ in the affine Weyl group W of G (because Γ contains representatives of the whole linear Weyl group W_0).

If two points in $\mathcal{C}$ are equivalent under Γ, they are at the same distance from ϕ. This can be proved first for SL_n, since j multiplies the distances by a constant. But in that case 1.3 can be proved directly. In fact, if $\underline{G} = SL_n$, $\gamma = (g_{ij})$, $\tau = \mathrm{diag}(t_i)$, $\tau^{-1}x = gx = y$, $x, y \in \mathcal{Q}$, then $\tau g \in P_x$, so

$$0 \leq \omega(t_i g_{ij}) + a_j(x) - a_i(x),\quad 1 \leq i,\ j \leq n.$$

If $\omega(t_{i_0}) < 0$, with $j \leq i_0 \leq i$, and $g_{ij} \neq 0$, we deduce from

$$a_{i_0}(y) - a_i(y) = a_{i_0}(x) - a_i(x) + \omega(t_i) - \omega(t_{i_0}) \text{ (cf. [1], 10-2-5, ii))}$$

that

$$0 \leq \omega(t_{i_0}) + a_{i_0}(y) - a_i(y) - a_{i_0}(x) + a_j(x) < (a_{i_0}(y) - a_i(y)) + (a_j(x) - a_{i_0}(x)) \leq 0,$$

i. e. a contradiction. Thus, $\omega(t_i) = 0$, for every i, and $x = y$.

One concludes the general case by the remark that a non-trivial translation cannot respect the distance to ϕ of all the points of a chamber.

1.4

To prove $\mathcal{T} = \Gamma . \mathcal{C}$ we use 1.2, which proves that $\Gamma . \mathcal{C}$ is both open and closed in the connected (even contractible!) space $\mathcal{T}$.

2. GENERALIZED AMALGAMS, PRESENTATIONS

2.1 'Amalgams'

Let G be an abstract group, $(G_i)_{i \in I}$ a family of subgroups of G.

Definition (see [11], [12], [13]). The sum of the groups G_i, $i \in I$, amalgamated on their intersections (shortly the 'amalgam' of the G_i's) is the inductive limit $\tilde{G}$ of the system of maps $G_i \cap G_j \to G_i$, $(i, j) \in I^2$.

In terms of presentations, it means that a presentation of $\tilde{G}$ can be obtained by taking for generators the union (in G) of sets of generators of the G_i's, submitted to the relations defining each of these groups (see [12], [13]).

'Amalgams' can also be characterized geometrically. Let X be a simplicial complex acted on by G (on left) in such a way that there exists a simplicial fundamental domain $X' \subset X$ (with the same meaning as in Theorem 1)

Theorem 2. Under the above hypothesis, if X' is connected and $x_0 \in X'$, there exists an exact sequence:

$$\pi_1(X, x_0) \to G \to \tilde{G} \to \pi_0(X) \to \{1\}.$$

The map on the left is injective whenever X' is simply connected, and each complex $gX' \cap X'$, $g \in G$, is connected.

Sketch of the proof (see also [12]). One first proves that $\tilde{G}$ is generated by the elements $g \in G$ such that $gX' \cap X'$ is not empty, with the relations $(gh) = g.h$ whenever $ghX' \cap gX' \cap X'$ is not empty. The theorem is then essentially due to the Macbeath-Weil theorem ([6], [17]). The injectivity of the map $\pi_1(X) \to \tilde{G}$ is obtained by looking at the proof of this theorem: either one describes the universal covering of X by gluing together the sets gX', or one compares the homotopy type of X with the one of the nerve of the family of subcomplexes $(gX')_{g \in G}$. q. e. d.

When G and the G_i's are given, such a space X can be defined as follows: take the nerve of the set of cosets gG_i, $i \in I$, $g \in G$.

2.2 The group $G(k[t])$ is an 'amalgam'

We use here the notations of paragraph 1. When $I \subset S$ is a set of simple roots, let Γ_I be the group generated by the elements $x_a(u)$, where $u \in k$ if $a \in I \cup (-I)$, $u = 0$ if $a \in -(S/I)$, and $u \in k[t]$ if $a \in S/I$.

Theorem 3. The group $\Gamma = G(k[t])$ is the sum of its subgroups Γ_I, $I \subset S$, amalgamated on their intersections.

Proof. Apply Theorems 1 and 2 (with $X = \mathcal{T}$ and $X' = \mathcal{Q}$), and the fact that Γ_I is the (filtering) union of the groups Γ_x such that $a(x) = 0$ iff a is in I. q. e. d.

In [10], U. Rehmann used this when he proved that, when k is finite and $\mathrm{rk}\, G \geq 3$, the group Γ is finitely presented. G. Harder has achieved in [5] a reduction theory for the action on buildings of groups over a ring of functions on a curve (defined over a finite field); maybe this could be used to study the finite presentation problem for this type of group.

2.3 Other examples of 'amalgams'

(a) Let K be a global field, S a set of finite valuations of K, $\mathcal{O}_S$ (resp. $\mathcal{O} = \mathcal{O}_\phi$) the ring of S-integers (resp. integers) in K.

Let $\underline{G}$ be as above a Chevalley simple and simply-connected group over $\mathbb{Z}$.

Theorem 4. *The group $\underline{G}(\mathcal{O}_S)$ is the 'amalgam' of a finite number of arithmetic groups over $\mathcal{O}$. If $G = SL_n$, $SL_n(\mathcal{O}_S)$ is the sum of $n^{\text{card } S}$ copies of $SL_n(\mathcal{O})$, amalgamated on their intersections.*

Proof. Let $v \in S$, K_v the completion of K with respect to v, $\mathcal{T}_v$ the Bruhat-Tits building of $\underline{G}(K_v)$, and X the product of the buildings $\mathcal{T}_v$, $v \in S$. The group $\underline{G}(\mathcal{O}_S)$ acts on X, and we shall prove that the product $X' = \prod_v \mathcal{C}_v$ of the fundamental affine chambers in $\mathcal{T}_v$ is a simplicial fundamental domain for this action. This is true for the group $\prod_v \underline{G}(K_v)$, but the isotropy groups are open in this product, and $\underline{G}(\mathcal{O}_S)$ is dense in it (by the approximation theorem). It then remains to describe the stabilizers of the vertices: these are arithmetic groups in general, and copies of $SL_n(\mathcal{O})$ when $\underline{G} = SL_n$. q. e. d.

One can deduce from this theorem explicit presentations of $\underline{G}(\mathcal{O}_S)$, starting with presentations of the arithmetic groups. The main difficulty is to find generators for the intersection of two of the groups one amalgamates.

(b) If $\underline{G}$ is a Chevalley group (scheme) as above and A a commutative ring with unit, the Steinberg group $St(A)$ associated to $\underline{G}(A)$ is defined by generators $x_a(u)$, $a \in \phi$, $u \in A$, subjected to the relations $x_a(u)x_a(v) = x_a(u + v)$ and (if $rk\underline{G} \geq 2$)

$$[x_a(u),\ x_b(v)] = \prod_{i,j} x_{ia+jb}(C_{i,j,a,b}u^i v^j) \quad (a + b \neq 0,\ i,\ j > 0),$$

where the C's are integral constants of $\underline{G}$. This shows easily that $St(A)$ *is the 'amalgam' of the unipotent radical of the parabolic subgroups of $\underline{G}(A)$ containing the standard torus* (see [2] for the quasi-split case). The Theorem 2 is then an interpretation of the definition of algebraic K-theory given by Volodin and Wagoner (see [15]). The exact sequence considered there gives (after stabilisation) the well-known exact sequence below:

$$\{1\} \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to \{1\}.$$

The analogue of this exists for the topological K-theory of a local ring [16].

3. HOMOLOGY

The notations are the ones of paragraph 1. We try to give an 'unstable homological analogue' of the homotopy invariance of algebraic K-theory: $K_*(k[t]) = K_*(k)$.

Theorem 5. *If the field* k *has a positive characteristic* p, *and if* F *is a field of coefficients with characteristic prime to* p, *the map* $H_*(\underline{G}(k); F) \to H_*(\underline{G}(k[t])); F$ *is an isomorphism.*

Proof. The action of $\underline{G}(k[t])$ on $\mathcal{T}$, described in Theorem 1, yields a classical spectral sequence converging to the homology of $\underline{G}(k[t])$ whose first term is the following:

$$E^1_{r,s} = \bigoplus_{\substack{\dim \sigma = r \\ \sigma \subset \mathcal{Q}}} H_s(\Gamma_\sigma, F),$$

where Γ_σ is the isotropy group of the simplex σ of $\mathcal{Q}$. The differential is given by alternative sums of corestrictions. By 1.1, the group Γ_σ is the semi-direct product of a reductive group $Z_\sigma(k)$ included in $\underline{G}(k)$ and a unipotent group U_σ. The latter is obtained by successive extensions from vector groups on k, i.e. inductive limits of finite p-groups. Thus, an iterated use of the Hochschild-Serre spectral sequence will prove that

$$H_*(\Gamma_\sigma, F) \simeq H_*(\Gamma_\sigma \cap \underline{G}(k), F).$$

The ordered set of groups $\Gamma_\sigma \cap \underline{G}(k)$, $\sigma \subset \mathcal{Q}$, has a maximal element $\Gamma_\rho = \underline{G}(k)$. Therefore, the second term $E^2_{r,s}$ of the spectral sequence, which can be thought of as the homology of this ordered set in the locally constant (co)sheaf given by the groups $H_s(\Gamma_\sigma \cap \underline{G}(k), F)$, is the following:

$$E^2_{r,s} = \begin{cases} H_s(\underline{G}(k)) & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$ q. e. d.

Theorem 6. If k is a finite field of characteristic p, and $\underline{G} = SL$ or Sp, then $H_*(\underline{G}(k[t]; \underset{\sim}{Z}/p\underset{\sim}{Z}) = 0$.

Proof. Let $k = F_q$ with $q = p^d$. We shall first prove that $H^i(\underline{G}(k[t])); \mathbb{Z}/p\mathbb{Z}) = 0$ when $0 < i < \mathrm{Inf}(n/2, d(p-1))$ (resp. $0 < i < d(p-1)/2$) when $\underline{G} = SL_n$ (resp. Sp_n). For this we use the spectral sequence described above in Theorem 5. When σ is a cell in $\mathcal{Q}$, a p-Sylow subgroup U'_σ of Γ_σ is generated by elements $x_a(u)$, with $a > 0$ and the degree of u bounded. It is normalized in Γ_σ by the standard maximal torus $\underline{T}(k)$ of $\underline{G}(k)$. So, it is sufficient to prove that $H^0(\underline{T}(k), H^i(U'_\sigma, \underset{\sim}{Z}/p\underset{\sim}{Z})) = 0$, for any $\sigma \subset \mathcal{Q}$, when i is bounded as above. Such a result can be proved first for GL_n as in [3], Prop. 4. 2. The case of GL_n implies the result for SL_n by the argument of [4] Prop. 5, and for Sp_n by [3], Lemma 4. 3.

Now to prove the theorem one interprets $H_*(\underline{G}(k[t]))$ in terms of characteristic classes and uses finite extensions of k as in [7] and [3], q. e. d.

Remarks. Maybe the same result could be proved for the Spin group too.

D. Quillen [8] proved results similar to Theorems 1, 3 and 5 for GL_n, using a building different from the Bruhat-Tits one. For GL, Theorem 6 is announced in [9], from a preprint of Gersten.

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List of problems

Edited by C. T. C. WALL

About sixty problems were suggested at the symposium, and a preliminary version of this list was circulated late in 1977. I am grateful for the large number of comments, including solutions of several of the problems, received from the participants as a result of the circulation. To avoid confusion, I have adhered to the numbering on the original list.

The problems are arranged (somewhat arbitrarily) in seven groups

A Algebraic K-theory
B Lengths and presentations
C Cohomology and algebraic groups
D 2-dimensional complexes
E Euler characteristics
F Finiteness conditions
G Group actions.

References are collected at the end of each group.

ALGEBRAIC K-THEORY

A0. The general problem is to obtain some methods of calculation of the groups $K_*(\mathbb{Z}\Gamma)$, $L_*(\mathbb{Z}\Gamma)$ for infinite groups Γ, where virtually nothing is known. Of particular interest is the homomorphism

$$\lambda_*(\Gamma)\colon h_*(B\Gamma; \underline{\underline{K}}_{\mathbb{Z}}) \to K_*(\mathbb{Z}\Gamma)$$

defined by Loday (1976), where $h_*(-; \underline{\underline{K}}_{\mathbb{Z}})$ is the generalised homology theory associated with the algebraic K-theory of $\mathbb{Z}$.

One may conjecture that $\lambda_*(\Gamma)$, or at least $\lambda_*(\Gamma) \otimes \mathbb{Q}$, is an isomorphism whenever Γ has type FP, or for some subclass (type FL, f. p. and type FL, Poincaré duality groups). It follows from results of Waldhausen (1976) that this property of Γ is inherited by amalgamated

free products. A recent result of Farrell and Hsiang (1978) shows that for Γ a Bieberbach group we have isomorphisms onto K_0, K_1. Some other results also are known in low dimensions (Loday, 1976).

Essentially the same remarks apply to L-theory; here a map analogous to $\lambda_*(\Gamma)$ has been defined by Ranicki (unpublished), and Cappell's splitting theorem (1971) gives a result (modulo, on occasion, 2-torsion) for amalgamated free products. Farrell and Hsiang (1978) likewise applies to L-theory (all dimensions) modulo reservations about 2-torsion (recently resolved).

A1. Find an explicit example of a torsion-free group Γ for which $\tilde{K}_0(\mathbb{Z}\Gamma)$ is nontrivial. Precisely one non-free projective module is known (Dunwoody, 1972) and that is stably free.

A2. If Γ is torsion-free, can $\mathbb{Z}\Gamma$ have zero-divisors?

A3. Does every finite group with periodic cohomology have a free resolution of minimum period? I showed (Wall, 1978) that in all cases twice the cohomology period suffices. The doubtful cases are those with subgroups

$$Q(8a;\, p,\, 1) = \langle x,\, y,\, t \,|\, x^{4a} = y^p = 1,\, t^2 = x^{2a},\, t^{-1}xt = x^{-1},$$
$$x^{-1}yx = y^{-1},\, t^{-1}yt = y\rangle$$

with $a > 1$ and $p > 1$ odd. All these have period 4.

Very recently it has been shown by Milgram (1978) that $Q(24;\, p,\, 1)$ does not have a free resolution of period 4 for $p = 5$, 7 or 11; though it does if $p \equiv 13 \pmod{24}$.

A4. Can one find a group π, of order N, with periodic cohomology and r prime to N such that the projective $\mathbb{Z}\pi$-module $\langle r, \Sigma\rangle$ is stably free but not free?

A5. The prime graph of a finite group π of order N has vertices the primes p dividing N; p and q are joined by an edge if π has an element of order pq. How does the number of components depend on π? If π is soluble, there are at most 2 components (see Gruenberg's

article in these proceedings); in all known examples, there are at most 6. Is this the most possible?

A6. Can one say anything about Whitehead groups of 1-relator groups; are there any elements besides units? or indeed (in the torsion free case) any nontrivial units?

A7. Various results in higher algebraic K-theory depend on detailed cohomological information; e.g. the question whether the classifying spaces of $GL_n(\mathbb{C})$ as topological and as discrete group have the same cohomology is related to the Lichtenbaum conjectures. More naively, it is known that

$$H^*(GL_n(\mathbb{Z}/p^2); \mathbb{Z}/l) \cong H^*(GL_n(\mathbb{Z}/p); \mathbb{Z}/l)$$

for $p \nmid l$: what happens if $p = l$?

References for Section A

Several lists of problems in this area exist already in the literature, and inevitably overlap the above. See particularly Bass (1973) and Shaneson (1973) and for wider lists to which I have contributed Kato (1975) and Browder (1976).

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LENGTHS AND PRESENTATIONS

B1. Let P be a pregroup in the sense of Stallings (1972), U(P) its universal group, $l : U(P) \to \mathbb{N}$ the natural length function. Does this satisfy the axioms of Lyndon (1963) (see also (Chiswell, 1976))? Chiswell states that the result is at least 'nearly true'!

B2. Describe the structure of a group with a real-valued length function satisfying Lyndon's axioms.

B3. Can every group with an $\mathbb{N}$-valued length function be embedded (length-preserving) in another such group such that each element is a product of elements of length one? Yes (Chiswell, 1979).

B4. Which finite groups have presentations with equal numbers of generators and relations? I wondered if this was related to periodicity of cohomology. The article by Johnson and Robertson in this volume gives enough examples to contradict naive guesses, but the property remains mysterious. One may also ask which finite groups admit cyclic presentations $\langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$ where r_k is obtained from r_1 by a suitable cyclic permutation of $x_1, \ldots, x_n$.

B5. Let $\langle X|R\rangle$ be a presentation satisfying a small cancellation condition. Does the normal closure of R in the free group $\langle X\rangle$ have a basis consisting of conjugates of elements of R? This is true for Fuchsian groups (Zieschang et al. (1970), p. 103).

B6. Let $\langle x_1, \ldots, x_n | r_1, \ldots, r_n\rangle$ be a cyclic presentation of the trivial group. Does it follow that r_1 coincides (modulo commutators) with some x_i or x_i^{-1}? A negative answer would entail producing a nontrivial element of the Whitehead group $Wh(\mathbb{Z}/n\mathbb{Z})$ which is realised by the presentation.

B7. Let $\underline{\underline{P}}_1 = \langle x|r(x)\rangle$, $\underline{\underline{P}}_2 = \langle y|s(y)\rangle$ be presentations with n generators and n relators of groups P_1, P_2. Define $\underline{\underline{P}}_1 \# \underline{\underline{P}}_2$ by substitution as $\langle y|r(s(y))\rangle$: let this present P. Then P_2 is a quotient of P, so if P is trivial, P_2 also is. Does it follow that P_1 is trivial too? If P_1 has a nontrivial finite dimensional representation (or equivalently, a proper subgroup of finite index) it follows from the techniques of Gerstenhaber and Rothaus (1962) that P is not trivial.

B8. Give explicit presentations of interesting arithmetic groups - see e.g. Swan (1971), Vinberg (1972), Behr and Mennicke (1968), Behr (1975a, b), Grunewald et al. (1978).

Similarly, find presentations for $GL_n(R)$, $SL_n(R)$ and $E_n(R)$ for interesting rings R, e.g. by adding relations to the Steinberg group $St_n(R)$, see e.g. Sylvester (1973). Similarly in the presence of a sesquilinear form, c.f. Sharpe (1972).

B9. A finite graph Y is called O-symmetric if Y is the Cayley graph of a group presentation $\underline{\underline{P}}$, and P is the automorphism group of Y. Classify all trivalent O-symmetric graphs.

For which $a, b, c \in \mathbb{Z}$ is the Cayley graph of

$$\langle R, S|R^2 = RS^aRS^bRS^c = 1\rangle$$

O-symmetric?

B10. Is the Burau representation of the braid group on $n \geq 4$ strings

faithful? See section 3.3 of Birman (1975) for definitions and detailed discussion. Related problems are listed in the appendix to that book.

B11. Suppose the finitely generated group G expressed in two ways as the fundamental group of a minimal graph of groups, such that each edge group is finite and each vertex group has at most one end. Is there a bijection between edges of the two graphs such that corresponding edge groups are conjugate? See Lemma 7.6 of the article by Scott and Wall in this volume.

B12. Let G be finite and $\mathfrak{g}$ its (integral) augmentation ideal. Call a subgroup H of G isolated (in G) if (i) $h^{-1}Hg \cap H$ equals 1 or H for every $g \in G$ and (ii) the centraliser in G of every nontrivial element of H is contained in H.

If $\mathfrak{g}$ decomposes, does G possess an isolated subgroup? This is true if G is soluble. The converse is true generally (Gruenberg et al. (1975)).

It is known that if $\mathfrak{g}$ decomposes, then the prime graph of G is disconnected (cf. problem A5). Is the converse true?

If the prime graph of G is not connected, does G possess an isolated subgroup?

B13. Let $E = G * H$ and let $\mathfrak{u}$, $\mathfrak{g}$, $\mathfrak{f}$ denote the augmentation ideals of E, G, H. If I is a right ideal of $\mathbb{Z}E$, let $d_E(I)$ denote the minimum number of generators of I as right ideal. Assuming G, H finitely generated, is it true that

$$d_E(\mathfrak{u}) = d_E(\mathfrak{g}E) + d_E(\mathfrak{f}E)?$$

(This would be a module analogue of the equality $d(E) = d(G) + d(H)$ which follows from the Gruško-Neumann theorem.)

In connexion with the above, given any group E and a finitely generated subgroup G, when can we expect $d_G(\mathfrak{g})$ to equal $d_E(\mathfrak{g}E)$?

B14. With $E = G * H$, as in B13, let E^* be the 'Cartesian subgroup' of E (i.e. the kernel of $E \to G \times H$). What is $d_E(E^*)$, the minimum number of generators of E^* as normal subgroup of E? (An obvious upper bound

is d(G)d(H).)

B15. Find examples of a group G, preferably finite, with a free presentation $F/R \simeq G$ such that $d_F(R) > d_G(R/R')$.

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COHOMOLOGY AND ALGEBRAIC GROUPS

C1. Let $\Gamma_0 = \langle a, t | t^{-1}a^2t = a^3 \rangle$. This is the famous non-hopfian 1-relator group of Baumslag and Solitar (1962). It is an HNN group with $\mathrm{cd}_{\mathbb{Z}}\Gamma = 2$, a duality group, but not residually finite.

Write $\Gamma_1 = [\Gamma_0, \Gamma_0]$ for the first, $\Gamma_2 = [\Gamma_1, \Gamma_1]$ for the second commutator subgroup and $\Gamma = \Gamma_0/\Gamma_2$ for the metabelian quotient. Since Γ_1 is the normal closure of $\langle a \rangle$, one can identify Γ_1/Γ_2 with $\mathbb{Z}[\frac{1}{6}]$; Γ is the obvious HNN group. The (corrected) problem was: is c. d. $\Gamma = 2$ or $= 3$? This has been answered by D. Gildenhuys, using the result of Bieri and Strebel (1978) that Γ is not finitely presented. In fact, c. d. $\Gamma = 3$ (Math. Zeits. 166 (1979), 21-5).

For any soluble torsion-free group Γ, the homological dimension equals the Hirsch number $h(\Gamma)$ and c. d. Γ equals either $h(\Gamma)$ or $1 + h(\Gamma)$: the former for Γ of type FP. In this (perhaps typical) example, we now have the second alternative.

C2. Given a short exact sequence $\Gamma' \to \Gamma \to \Gamma''$ of groups of type FP, is it true that c. d. Γ = c. d. Γ' + c. d. Γ''? The answer is yes (Feldman) for groups of type FP over a field: see Bieri (1976) p. 70. Over $\mathbb{Z}$, if $m = \mathrm{c.d.}_{\mathbb{Z}}\Gamma'$, $n = \mathrm{c.d.}_{\mathbb{Z}}\Gamma''$ we have

$$H^{m+n}(\Gamma; \mathbb{Z}\Gamma) \cong H^m(\Gamma'; \mathbb{Z}\Gamma') \otimes_{\mathbb{Z}} H^n(\Gamma''; \mathbb{Z}\Gamma'').$$

C3. Is every group Γ with c. d. $\Gamma = 2$ the fundamental group of a graph of free groups? An alternative (perhaps more plausible) version is: let $\mathcal{C}_{1,0}$ denote the class of free groups and (inductively) $\mathcal{C}_{1,n}$ the class of groups $G = H *_F K$ or $H *_F$ with H, K in $\mathcal{C}_{1,n-1}$ and F

free. Let $\mathcal{C}_2 = \cup_{n=0}^{\infty} \mathcal{C}_{1,n}$. Does every (f. g.) group Γ with c. d. $\Gamma \leq 2$ belong to $\mathcal{C}_2$?

The answer to C1 adds credence to this conjecture.

C4. Suppose v. c. d. $\Gamma < \infty$ (and, if necessary, that Γ has type VFP). Is the Farrell (1977) cohomology $\hat{H}^*(\Gamma)$ annihilated by the l. c. m. of orders of finite subgroups of Γ? See Ken Brown's paper in these proceedings for background.

C5. Is the dualising module of a duality group always $\mathbb{Z}$-free? More generally, is $H^*(\Gamma, \mathbb{Z}\Gamma)$ $\mathbb{Z}$-free for any Γ of type FP? The stronger result would decide C2.

C6. Is there a simple Poincaré duality group?

C7. Bousfield's (1977) $\mathbb{Q}$-completion $\hat{G}_{\mathbb{Q}}$ of a group G may be defined as follows. Write $G_{(0)} = G$, $G_{(n+1)} = [G, G_{(n)}]$ for the lower central series, and $(G/G_{(n)})\hat{}$ for the Mal'cer (1949) completion of the nilpotent group $G/G_{(n)}$. Then take the inverse limit $\hat{G}_{\mathbb{Q}} = \lim_{\leftarrow n} (G/G_{(n)})\hat{}$. For F finitely generated free, does $H_2(\hat{F}_{\mathbb{Q}}; \mathbb{Q})$ vanish? This is important for the theory of such completions. There is an analogous question with $\mathbb{Z}/n$ replacing $\mathbb{Q}$.

C8. A simplicial complex is said to have the property CM if both it, and the link of any vertex in it, are homotopy equivalent to bouquets of spheres of constant dimension. Now let Γ be of type VFL, $\underline{\underline{A}}_p(\Gamma)$ the partly ordered set of elementary abelian p-subgroups of Γ, under inclusion. Are the following two concepts related? (i) The realisation $|\underline{\underline{A}}_p(\Gamma)|$ is a CM complex. (ii) $H^*(\Gamma; \mathbb{Z}/p)$ is a Cohen-Macaulay ring.

C9. Let G be a quasi-simple algebraic group over $\mathbb{Q}$, $\hat{G}(\mathbb{Q})$ the profinite completion of $G(\mathbb{Q})$ (here the topology is defined by arithmetic subgroups), $G(\hat{\mathbb{Q}})$ the adelic completion (topology of congruence subgroups), $C^S(G) = \mathrm{Ker}(\hat{G}(\mathbb{Q}) \to G(\hat{\mathbb{Q}}))$. Compute $C^S(G)$. This is the well-known congruence subgroup problem.

The answer is known (Bass et al. (1967)) for SL_n and Sp_n of a

commutative global field K; this has been extended (Matsumoto, 1969) to all split groups, (Kneser, 1969) to the orthogonal group of any quadratic form of Witt index ≥ 2 over K, and (Bak, 1978: see Theorem 1.79) to the unitary group of a hyperbolic hermitian form of rank ≥ 4 defined by a quadratic extension L/K. Raghunathan (1976) has shown that $C^S(G)$ is finite when $\mathbb{Q}$-rank $G \geq 2$.

In all cases so far, if G has field of definition K, and index (in some sense) ≥ 2, $C^S(G)$ is trivial if K is not totally complex; otherwise is isomorphic to the group $\mu(K)$ of roots of unity in K. One may conjecture that this holds in general. The key outstanding cases are linear etc. groups over division rings.

C10. Let G be a semisimple (or reductive) algebraic group over $\mathbb{Q}$. I originally asked: are maximal p-subgroups of $G(\mathbb{Q})$ conjugate? This was proved for $GL_n(\mathbb{Q})$ by Volvacev (1963). However, Serre points out that though Sylow 3-subgroups of $SL_2(\mathbb{Q})$ (of order 3) are conjugate in $GL_2(\mathbb{Q})$, A and $g^{-1}Ag$ are conjugate in $SL_2(\mathbb{Q})$ only if $\pm\det g$ is of the form $a^2 + 3b^2$ $(a, b \in \mathbb{Q})$. Thus there are infinitely many classes.

The problem of obtaining a better understanding remains. Often (e.g. in $GL_n(\mathbb{Q})$ with $p \neq 2$) we can find a prime $l \neq p$ such that a maximal p-subgroup of $G(\mathbb{Q})$ is also a maximal p-subgroup of $G(\hat{\mathbb{Q}}_l)$, where $\hat{\mathbb{Q}}_l$ denotes the field of l-adic numbers. It is then contained in a maximal compact subgroup (for which Sylow theory holds), and though these are not all conjugate, they fall into a finite number of classes (Bruhat and Tits (1972) p. 65).

Indeed if G is reductive over $\hat{\mathbb{Q}}_p$, it can be shown that finite subgroups of $G(\hat{\mathbb{Q}}_p)$ fall into a finite number of conjugacy classes. Thus if G is reductive over $\mathbb{Q}$, the finite subgroups of $G(\mathbb{Q})$ are finite in number modulo conjugation by any $G(\hat{\mathbb{Q}}_p)$.

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2-DIMENSIONAL COMPLEXES

It is easy to find unsolved problems here. Most of them are generated by the following.

If the connected, finite 2-complexes K and K' are equivalent by 2- and 3-moves, then they are simple homotopy equivalent; this in turn implies homotopy equivalence, and hence that we have isomorphic fundamental groups and equal Euler characteristics. Which of these implications

can be reversed? - or more precisely, under what conditions can the various implications be reversed? The first is unknown (Andrews and Curtis (1965) conjecture) even in the contractible case.

We can also generalise the question by allowing K, K' to have a common subcomplex L, with $\dim(K - L) \leq 2$, $\dim(K' - L) \leq 2$, and working relative to L throughout. In view of the close connexion between group presentations and 2-complexes, it is not too difficult to translate these questions into group theoretic terms.

Along with these 'uniqueness' questions are companion 'existence' ones: here (K, L) is given (satisfying some conditions) and we seek (K', L), equivalent to (K, L) in an appropriate sense, and with $\dim(K' - L) \leq 2$. A number of positive results have appeared recently on 'stabilising' by replacing K, K' by bouquets with copies of S^2.

We now list some of the more interesting questions explicitly.

D1. Suppose Y dominated by a connected 2-complex. Is Y homotopy equivalent to a 2-complex? For Y satisfying condition F2 of Wall (1965) this seems improbable. Ratcliffe has shown (unpublished) that if $G = \pi_1(Y)$ is f.p., and over the group ring $\mathbb{Z}G$ 'big projective modules are free', then if Y does <u>not</u> satisfy F2 it is indeed homotopy equivalent to the bouquet of infinitely many 2-spheres with X, X any finite 2-complex with fundamental group G.

D2. In Wall (1965) a construction was given for a space X dominated by a finite 2-complex but not homotopy equivalent to one: X can be taken as a 3-dimensional CW-complex. One cannot take X to be a compact ANR (West, 1977), but can X be a 2-dimensional compactum? The constructions of Ferry (1978) show that X can be a 3-dimensional compactum.

D3. Now suppose Y dominated by a finite connected 2-complex and that the Wall obstruction vanishes. Does it follow that Y is homotopy equivalent to a finite 2-complex (say for short, Y is finite)? It has been shown by several people (Dyer (1975), Cohen (1977), Ratcliffe, Schafer) that some bouquet of Y and a finite number of 2-spheres is finite. Also (Dyer, 1975) if $\pi_1(Y)$ is finite abelian and $\chi(Y)$ not minimal, then Y is finite. The same conclusion follows for any finite group $\pi_1(Y)$ from Browning

(1978) and Dyer (1978a). Also, Y is always finite if $\pi_1(Y)$ is cyclic (Dyer (1975) or Cockroft et al. (1975)), free (follows from Wall (1965) using Stallings (1968)) or the product of two finite cyclic groups (Dyer, 1978b).

J. Cohen conjectures that $Y \vee S^2$ is always finite, and that Y is if either $\pi_1(Y)$ is infinite or $\pi_2(Y)$ a Swan module over $\mathbb{Z}\pi_1(Y)$ (a Swan module can be generated by k elements if each of its localisations can).

D4. Does $c.d.\Gamma = 2$ imply the existence of a 2-dimensional $K(\Gamma, 1)$? (Conjecture of Eilenberg and Ganea (1957).) This is a special case of D1; there is also a finite version as in D3. The problem can be formulated in terms of the relation module of a presentation. It would be settled by an affirmative answer to C3. One approach is to try to prove Γ decomposable as free product (or HNN group) with amalgamated free subgroup, and arrive eventually at simpler (hopefully free) indecomposable groups.

D5. For Γ an f.p. group, a Γ-complex is a connected finite 2-complex with fundamental group Γ. Call the Γ-complex X a <u>root</u> if it is not homotopy equivalent to $Y \vee S^2$ for some Γ-complex Y, and set

$$\text{level}(X) = \chi(X) - \min\{\chi(Y) : Y \text{ a } \Gamma\text{-complex}\}.$$

At what levels can roots occur? For Γ finite, roots only occur at level 0 (Browning, 1978). The example of Dunwoody (1972) (see references for A), with Γ the trefoil group, gives a root of level 1. For further discussion, see Dyer's article in these proceedings.

D6. Are homotopy equivalent finite 2-complexes always simply homotopy equivalent? This holds for cyclic groups (Dyer and Sieradski, 1973) Cockroft and Moss, 1975); indeed, for all finite abelian π with $SK_1(\mathbb{Z}\pi) = 0$ (Dyer, 1978b); also stably, and if π is finite, at most two 2-spheres are needed.

D7. Is a subcomplex of an aspherical 2-complex aspherical? (Whitehead conjecture.) Equivalently, can one ever kill π_2 of a 2-complex by attaching 2-cells? Partial results have been obtained by Cockroft (1954), Adams (1955) and Cohen (1978), in the case when the fundamental group of the subcomplex has no perfect subgroup.

D8. Characterise the set of elements $x \in Wh(\Gamma)$ which are torsions of inclusions $i : X \subset Y$ where i is a homotopy equivalence and $\dim(Y-X)=2$. Is this set a subgroup? Results of Rothaus (1977) show that for Γ dihedral we may not obtain all elements of $Wh(\Gamma)$. A negative answer to problem B6 would give a non-trivial example here.

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EULER CHARACTERISTICS

In the following, I will use the notation from Bass' article in this volume: Thus if M is projective, or of type FP as A-module, r_M denotes the trace of the identity map of M; when $A = k\Gamma$ this may be regarded as a k-valued function on conjugacy classes of Γ. In particular, if $A = k\Gamma$ and $M = k$, so that Γ is of type FP over k (I will write FP_k), write $\chi_\Gamma^k = \chi_\Gamma$ for r_k and $\chi(\Gamma) = \chi_\Gamma(1)$ for the Euler characteristic; $\Sigma(\chi_\Gamma)$ for the sum over conjugacy classes, or homological Euler characteristic.

E1. Here we repeat problems from 4.4 and 7.6 of Bass' article. Suppose first $k \subset \mathbb{C}$ such that $k \cap \mathbb{Q} = \mathbb{Z}$, and P f.g. projective over $k\Gamma$.

Weak conjecture: $r_P(1) = \Sigma r_P(\tau)$

Strong conjecture: $r_P(s) = 0$ for all $s \neq 1$.

For Γ of type $FP_{\mathbb{Z}}$, the weak conjecture implies that $\chi(\Gamma) = \Sigma(\chi_\Gamma)$ the strong conjecture that $\chi_\Gamma(s) = 0$ for all $s \neq 1$.

For Γ of type $FP_{\mathbb{Q}}$, it is further conjectured that $\chi_\Gamma(s) = 0$ for s of infinite order; that there are only finitely many classes of elements of finite order; and that if s has finite order and centraliser $Z_\Gamma(s)$ of type $FP_{\mathbb{Q}}$, then $\chi_\Gamma(s) = \chi(Z_\Gamma(s))$.

The conjectures in the preceding paragraph have been established by

Ken Brown (1978), but under rather complicated hypotheses. These include, for example, the case of arithmetic groups, or of S-arithmetic groups in reductive algebraic groups. Things would simplify considerably if the strong conjecture could be proved in general.

E2. For Γ of type $FP_{\mathbb{Q}}$ with $\chi_\Gamma(1) \neq 0$, does it follow that $\chi_\Gamma(s) \neq 0$ for all s of finite order in Γ? The answer is no, and this follows from (Brown, 1978): a particular counterexample (of type $Sp_{2n}(A)$) is given in (Brown, 1974).

E3. If p^n divides the denominator of $\chi(\Gamma)$, must Γ have a subgroup of order p^n? An affirmative answer to this, too, is given by (Brown, 1978) provided, for example, that every finite p-subgroup F of Γ has normaliser $N_\Gamma(F)$ of type VFP.

E4. Can one find a group Γ such that $\chi(\Gamma)$ has denominator $m > 1$, but all subgroups of finite index in Γ have index prime to m?

E5. Does a nontrivial torsion-free group of type $FP_{\mathbb{Q}}$ necessarily have infinitely many conjugacy classes?

E6. Suppose Γ of type FP over $\mathbb{F}_p$. Does it follow that

(i) Γ has type $FP_{\mathbb{Q}}$?

(ii) $\chi_\Gamma^{\mathbb{Q}}$ takes p-integral values?

(iii) The mod p reduction of $\chi_\Gamma^{\mathbb{Q}}$ is $\chi_\Gamma^{\mathbb{F}_p}$?

E7. Give a formula for $\chi_\Gamma(1) - \Sigma(\chi_\Gamma)$ for Γ of type $FP_{\mathbb{Q}}$. Such a formula, under suitable hypotheses, follows from the results of Brown (1978).

E8. J. Cohen has recently proposed a generalised definition of Euler characteristic. Can this be used to weaken the finiteness hypotheses in any of the results of Bass and Brown?

E9. In the early circulation, I reproduced conjectures 1 and 2 from Stallings (1974): in fact these (when precisely formulated) are largely known (see Bass' article). The following remains open: suppose Γ of

type FP_K, Γ' normal in K with $H_*(\Gamma'; K)$ f.g. and $c.d._K(\Gamma/\Gamma') < \infty$. Then is Γ/Γ' of type FP_K?

As to Stallings' further questions, Conjectures 3a and 3b reappear as E3 and E6 above; Problem 2a as E4, 2b as part of E1 and Problem 3 is solved by work of Bass and Brown. Problem 1 asks for 'reasonable conditions' for property FP_K to be hereditary for normal subgroups.

E10. Let $\phi(\Gamma)$ be a function on some class of groups Γ, such that for a subgroup $\Gamma' \subset \Gamma$ of index n, $\phi(\Gamma') \leq n\phi(\Gamma)$. Define

$$\tilde{\phi}(\Gamma) = \inf\{n^{-1}\phi(\Gamma') : \Gamma' \text{ a subgroup of finite index } n \text{ in } \Gamma\}.$$

Then $\tilde{\phi}(\Gamma') = n\tilde{\phi}(\Gamma)$, and $\tilde{\phi}$ is a generalised Euler characteristic. Examples: $g(\Gamma)$ = minimum number of generators for Γ, $R(\Gamma)$ = minimum (over all presentations) of total of lengths of relators.

Very little is known about $\tilde{g}(\Gamma)$ and $\tilde{R}(\Gamma)$: a first problem is to make some nontrivial calculations. Also, if M, N are hyperbolic 3-manifolds and $f : M \to N$ has degree d, is $\tilde{R}(\pi_1(M)) \leq |d|\tilde{R}(\pi_1(N))$?

This problem is related to work of Thurston (1978), particularly to §6.

E11. Suppose Γ of type VFP and s of finite order in Γ. Does it follow that the centraliser $Z_\Gamma(s)$ also has type VFP? An affirmative answer, with the results of Brown (1978), would imply that groups of type VFP have only finitely many conjugacy classes of elements of finite order.

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FINITENESS CONDITIONS

Perhaps the most interesting questions concern relations between different types of finiteness condition. These are in fact very diverse.

F1. Is every countable (not necessarily f. g.) torsion free group with infinitely many ends a free product? Does every uncountable locally finite group have 1 end? See the article by Scott and Wall for background.

F2. Which 1-relator groups are (nontrivial) amalgamated free products?

F3. We say that an f. g. group with at most one end is O-accessible, and that a group Γ is n-accessible if $\Gamma = A *_F$ or $A *_F B$ with F finite and A (and B) $(n-1)$-accessible. Is every f. g. group Γ n-accessible for some n? It was shown by Dunwoody (1978) that equivalent conditions are

(i) $H^1(\Gamma; R\Gamma)$ is f. g. as $R\Gamma$-module, or

(ii) $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} H^1(\Gamma; \mathbb{Z}\Gamma)$ is f. g. as abelian group.

F4. Does every discrete f. g. subgroup of a Lie group G have a torsion-free subgroup of finite index? The answer is no: counterexamples have been given by Deligne (1978), Millson and Serre. One may ask instead: does every connected semisimple Lie group G have a torsion-free subgroup Γ with G/Γ compact? This is known (Borel, 1963) if G has a faithful linear representation.

F5. Does every small cancellation group have a torsion-free subgroup of finite index (it will then be of type VFP)? Is it residually finite? For 1-relator groups, the first conclusion holds, by Fischer et al. (1972).

F6. Are all duality groups residually finite? Bieri points out that the group Γ_0 defined in C1 gives a counterexample. Indeed, Thurston can construct Poincaré duality groups which are not residually finite. One may ask in contrast (C6) whether there exists a simple Poincaré duality group.

I also asked when residual finiteness is inherited by amalgamated free products $A *_C B$, $A *_C$. This certainly does not hold in general.

The best (or the only?) positive result is when C is finite (Baumslag, 1963).

F7. In an opposite direction, one may seek conditions implying that elements of finite order, or finite subgroups, fall into finitely many conjugacy classes. Does this hold when $\text{v. c. d. } \Gamma < \infty$? when Γ is of type VFP? when Γ is of type $FP_{\mathbb{Q}}$?

The paper of Brown (1978) (see references for E) yields the conclusion when $\text{v. c. d. } \Gamma < \infty$ and for each $s \in \Gamma$ of finite order, $Z_{\Gamma}(s)$ has a torsion-free normal subgroup of finite index with finitely generated integral homology. Another (stringent) sufficient condition has been given by Howie and Schneebeli (preprint, ETH, Nov. 1978).

F8. Is every group of type $FP_{\mathbb{Z}}$ also of type FL? Does this at least hold for duality (or Poincaré duality) groups? Or under the assumption that the group has type VFL?

F9. Does every group of type $FP_{\mathbb{Q}}$ have a finite series with quotients of type $FP_{\mathbb{Z}}$ or finite? The answer is negative (Bieri). For suppose

(i) Γ of type $FP_{\mathbb{Q}}$, with $\text{c. d.}_{\mathbb{Q}}\Gamma = 2$,

(ii) Γ has no proper subgroup of finite index,

(iii) Γ is not torsion-free.

Then $\Gamma * \Gamma$ has no proper normal subgroup which is finite (it is a free product) or of finite index (Γ has none): hence (Bieri, 1978) if Γ' is a proper normal subgroup with quotient Γ'', $\text{c. d.}_{\mathbb{Q}}\Gamma'' = 1$. By Dunwoody (1978), Γ'' has a free subgroup of finite index: a contradiction.

Examples of Γ satisfying the conditions are given by Schneebeli (1978) (see the group G given for Theorem 1). Bieri remarks that one can also find a duality group as counterexample.

For the next three problems, we say that Γ has type FP_n $(n \leq \infty)$ if $\mathbb{Z}$ has a $\mathbb{Z}\Gamma$-projective resolution C_* such that C_k is finitely generated for $k \leq n$.

F10. Is every group of type FP_2 ('almost finitely presented') necessarily finitely presented? Does this at least hold for Poincaré duality groups?

F11. Is every torsion-free group of type FP_∞ also of type FP? Or, on the other hand, can one find an FP_∞ group with a non-finitely generated free abelian subgroup?

F12. Let C be a complete curve over a finite field, with function field E, S a set of n points of C, G a simple algebraic group over E with E-rank k. Is it true that any S-arithmetic subgroup of G(E) is virtually of type FP_{k+n-2} but not of type FP_{k+n-1}? See Behr's article in these proceedings for known results in this direction.

F13. Let Γ be an f.g. group, with a faithful finite-dimensional linear representation, such that the character of each finite-dimensional representation ρ takes values integral over $\mathbb{Z}$. Does it follow that Γ is an arithmetic group? As a first step, do the images $\rho(\Gamma)$ have Zariski closures of bounded dimension?

F14. Call a group coherent if every finitely generated subgroup is finitely presented. For example, this holds trivially for abelian groups, and (Scott, 1973) for fundamental groups of 3-manifolds. Are the following coherent?

(a) $SL_2(\mathbb{Z}[1/p])$, (b) $SL_3(\mathbb{Z})$,

(c) $\Gamma *_A \Gamma'$ where Γ, Γ' are free and A has finite index in each.

See Serre (1974) as an earlier reference, and for the observations that $SL_4(\mathbb{Z})$, $SL_2(\mathbb{Z}[1/pq])$ are not coherent. S. M. Gersten (preprint, Utah, May 1979) gives a negative answer to (c).

F15. Are small cancellation groups - or their group rings - coherent? Serre points out that the product of two free groups of rank 2 has presentation

$$\langle x, y, z, t \mid xzx^{-1}z^{-1} = xtx^{-1}t^{-1} = yzy^{-1}z^{-1} = yty^{-1}t^{-1} = 1 \rangle$$

satisfying C(4), T(4) of Lyndon and Schupp (1977), p. 240, so is a small cancellation group. On the other hand, it is not coherent.

F16. Let Γ be a Poincaré duality group of dimension ≥ 3. Is the 'fundamental group at infinity' of Γ necessarily trivial? This is known in

many cases, e.g. if Γ has an f.p. normal subgroup Γ' of infinite index and either Γ' or Γ/Γ' has one end. In dimensions ≥ 5 it is equivalent to having the universal cover of a compact $K(\Gamma, 1)$ manifold homeomorphic to euclidean space. See Johnson (1974, 1975), Lee and Raymond (1975).

F17. Are there any groups, other than finite extensions of f.g. nilpotent groups, with polynomial growth? See Milnor (1968) and Wolf (1968) for background.

F18. Let F be a finitely generated free group, Γ its automorphism group or outer automorphism group. Is $\mathrm{vcd}_{\mathbb{Z}}\Gamma < \infty$? This holds if F is the fundamental group of a closed surface, using the action on Teichmüller space.

F19. A group Γ, nilpotent of class n, is terminal if there is no group Γ', nilpotent of class $(n+1)$, with $\Gamma \cong \Gamma'/\Gamma'_{(n)}$ (notation as in C7). Evans (1968) gave a homological criterion for terminality. Can one find similar criteria for the following?

(a) there exists a terminal Γ' of some class $m > n$ with $\Gamma \cong \Gamma'/\Gamma'_{(m)}$

(b) there is a residually nilpotent (but not nilpotent) group Γ'' with $\Gamma \cong \Gamma''/\Gamma''_{(m)}$;

(c) for any $m > n$ there exists Γ' of class m with $\Gamma \cong \Gamma'/\Gamma'_{(m)}$.

F20. Let Γ be an f.g. group with a faithful finite dimensional representation over $\mathbb{C}$. Define tr. deg Γ to be the least transcendence degree over $\mathbb{Q}$ of a subfield F of $\mathbb{C}$ over which Γ has a faithful representation. Interpret tr. deg Γ group theoretically. Is it true that

$$\text{tr. deg } \Gamma = \max\{\text{tr. deg } \Gamma' : \Gamma' \subset \Gamma \text{ solvable}\} ?$$

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GROUP ACTIONS

A key general problem is to relate group cohomology properties to existence of group actions. Of particular interest here are

(a) If π is finite, with periodic cohomology, can it act freely on a

sphere? Of dimension one less than the period? The answers here are essentially known: see Madsen et al. (1976), Wall (1978); also under A2.
(b) If c. d. $\Gamma < \infty$, can Γ act freely and properly on euclidean space? Yes (Wall, 1970). If also Γ is a Poincaré duality group, is there such an action with compact quotient? This can be broken up into further questions see G2.
(c) One can formulate common generalisations of (a) and (b): see e. g. G4.

G1. Let Γ' be a subgroup of finite index in Γ, and suppose given a free, proper action of Γ' on $\mathbb{R}^n$ with compact quotient. Does the action necessarily extend to one of Γ? A slightly less general result with $n = 2$ was proved by Nielsen long ago.

In the case when Γ' is a Mostow-Wang group (extension of a f. g. torsion free nilpotent group by a f. g. free abelian group) and the action is standard (induced by embedding in a Lie group), the conjecture was proved by Auslander and Johnson (1976).

G2. Is every Poincaré duality group Γ the fundamental group of a closed $K(\Gamma, 1)$ manifold? Smooth manifold? Manifold unique up to homeo morphism? (It will not be unique up to diffeomorphism.)

In the surgery approach one needs to show first that Γ is f. p. (F10) and of type FL (F8), then that there exists a normal invariant - i. e. that certain obstructions in $H^{4k+3}(\Gamma; \mathbb{Z}/2)$, $H^{4k+1}(\Gamma; \mathbb{Z}_{(2)})$, $KO^*(K(\Gamma, 1)) \otimes \mathbb{Z}[\frac{1}{2}]$ vanish - and then study the surgery obstruction (Problem A0). If the obstructions do not vanish, what are they? In a practical attempt at making progress one will place further restrictions on Γ - e. g. Γ an extension of Γ' by Γ'' where suitable manifolds exist for Γ' and Γ'', or for Γ' with Γ'' finite (G1). The best results to date are those of Farrell and Hsiang (1978) (see references for A).

G3. Suppose Γ has a series whose quotients are fundamental groups of closed, orientable surfaces. Is Γ the fundamental group of a non-singular projective (complex algebraic) and aspherical variety? Johnson (1978) has shown that this is virtually true.

G4. Suppose Γ countable, vcdΓ finite, and the Farrell cohomology of Γ periodic. Is there a free proper action of Γ on some product $S^k \times \mathbb{R}^n$?

G5. Let G be a finite complex reflection group acting on V, X the union of the reflection hyperplanes. Compute $\pi_1(V - X)$, and decide whether $V - X$ is a $K(\pi, 1)$. For the complexification of a real reflection group, these were answered by Brieskorn (1971) and Deligne (1972). The case $\dim V = 2$ was analysed by Bannai (1976): she computed the groups, and gave equations from which one can deduce easily that $V - X$ is a $K(\pi, 1)$. The imprimitive case is also not difficult. As asphericity is inherited by products, one need only consider irreducible groups. There remain cases 24-27, 29, 31-34 of the list of Shephard and Todd (1954).

G6. Let $\text{vcd}\,\Gamma = n$. Is there a contractible n-complex on which Γ acts properly?

G7. Suppose $H_*(X; \mathbb{Z})$ f.g. and $\dim X < \infty$, and that Γ is a torsion free group acting properly on X with X/Γ compact. Does it follow that Γ has finite cohomological dimension?

G8. Let Γ be a torsion free group acting differentiably (C^1) on S^n, fixing a point P and acting properly on the complement of P. Must Γ be a Bieberbach group? See Kulkarni (1977) for background.

G9. Let G be an algebraic group acting on an algebraic variety X (all over $\mathbb{R}$), Γ a discrete Zariski-dense subgroup such that the action of Γ is proper. The original question was: must the action of G be proper? (this holds for unipotent groups). Both Serre and Kulkarni gave counter-examples: the simplest is $G = G_m \times G_m$ acting on G_m (the multiplicativ group of nonzero reals) by the second projection; Γ generated by $(2, 3)$. The following remains open. Can one find a simple Lie group G acting on a locally compact Hausdorff space X, and a subgroup Γ of G acting properly discontinuously on X with $\Gamma\backslash X$ compact, such that there is no closed subgroup H of G containing Γ, with $\pi_0(H)$ finite, which acts properly on X?

G10. Is there a finite CW complex K such that $\pi_n(K)$ vanishes for $n \geq n_0$ but K is not a $K(\pi, 1)$? Is there one with $n_0 = 3$?

G11. For any manifold W, with $\dim W \geq 5$, characterise the set

$$\underline{\underline{S}} = \{\pi_1(M) : M \text{ a manifold, } \phi : M \to W \text{ of degree } 1\}.$$

If there is a split epimorphism $G \to \pi_1(W)$, $G \in \underline{\underline{S}}$. If $G \in \underline{\underline{S}}$, then for any $\pi_1(W)$-module A, $H_1(G; A) \to H_1(\pi_1(W); A)$ is a split epimorphism.

G12. Find an analogue of Stallings' structure theorem for pairs (Γ, Γ') with $e(\Gamma) = 1$ but $e(\Gamma, \Gamma') \geq 2$. The example given in Chapter 8 of the article by Scott and Wall in this volume should be noted. Here A and C are infinite f. g. simple groups and $G = A * C$. One may now take $\Gamma = (A * C) \times C$, $\Gamma' = C \times C$: then $e(\Gamma) = 1$, $e(\Gamma, \Gamma') = e(G, C) = \infty$ (Lemma 8.2iii, loc. cit.) but there is no splitting.

G13. Does there exist a closed aspherical 4-manifold with negative Euler characteristic?

G14. Can one find a connected manifold M, homotopy equivalent to a 1-complex (with negative Euler characteristic), and a proper free action of a discrete group on M with compact quotient?

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